Geometry of black holes and braneworlds in higher dimensions

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Geometry of Black Holes and Braneworlds in Higher Dimensions

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A Thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Theory
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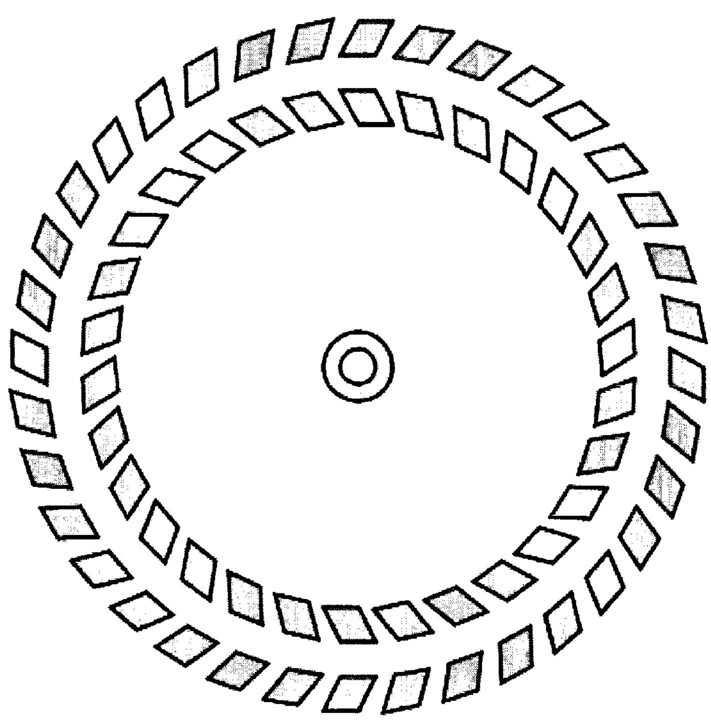
England

September 2004

Dedicated to

friends and family



The Pinna-Brelstaff Illusion

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Abstract

This thesis first discusses braneworld models, we explain how the bulk geometry in codimension 2 scenarios restricts braneworld fields in a way inconsistent with observation. We then show how generalising Einstein's equations to include Gauss-Bonnet terms avoids this problem and as an example we successfully reproduce the Friedmann-Robertson-Walker cosmology familiar in Einstein gravity. The work on braneworlds concludes with a detailed perturbation analysis of a simple conical space-time in Gauss-Bonnet gravity, non-trivially we find the standard four dimensional Lichnerowicz equation on the brane even though the calculation is performed in six dimensions. Next, motivated by the microscopic description of black hole thermodynamics, we discuss Gubser and Mitra's conjectured relationship between classical and thermodynamic stability including a review of numerical and theoretical evidence for it. We then give an argument using a recently discovered ansatz for non-uniform smeared p-brane solutions that the conjecture fails in the generality in which it is proposed. The thesis emphasises the underlying relationship between worldvolume field theory and bulk gravity from a geometrical point of view throughout.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and is all my own work unless referenced to the contrary in the text.

Chapter 2 of this thesis is a review of background material necessary to put later results into perspective. Chapter 3 contains original work done in collaboration with my supervisor Ruth Gregory, Ignacio Navarro and Jose Santiago [53], however section 3.7 onwards is my own work. Chapter 4 contains more necessary review material and Chapter 5 contains original work done in collaboration with Simon Ross [67].

Acknowledgements

I would first like to thank my family, regardless of how far away I have been you should all know that none of this would have been possible without you. I would also like to especially mention Giovanna Scataglini and Melissa Bordiere for reasons my words can't do justice, I will just hide behind

"帰ってきてくれ、でないと、俺、死ぬよ。.

Having been in Durham for eight years I have had the pleasure to make many friends, almost two years ago I was fortunate enough to rent Barbara's spare room, I can't thank her enough for all the kindness she has shown me - it won't be the same without you. I could probably write an essay on her drop out student, just a few words wouldn't do my gratitude justice, in any case I would like to thank Sharry for everything he has done for me, especially for what you never did or said.

It will be a loss not to be greeted every morning in the department, thank you Mary for your friendship over the last few years and for sharing your views and opinions with me. It would be a crime not to mention the Lone Fiasco you made me work many moons for 14/06! - good luck with everything, even if you did make me walk three miles in the pouring rain and cold when I wasn't feeling very well because you didn't want to have to sit an extra couple of minutes in the traffic on Potters Bank! - don't forget that good word if it is ever appropriate! - grazie. And last, but not least, I would like to say to Sharron Davis - I'm glad I never did bite my tongue off and whatever you do just be careful not to find yourself inside out!

Most of my time working here was spent with those who had to share an office with me; Jessica who put up with me for the whole four years, Adam for only two years but that odd shoe incident I will carry around with me for a long time yet and of course Anthony - who wisely moved to the IPPP building - thanks for trusting me enough to care for Tufty! It wouldn't have been the same without Sheila, you tried to keep me right for two years and I still think I should have kept a record of your life story, it would make a best seller! Not everyone is convinced about the Jesus look mind.

Finally I would like to thank the two people who most directly shaped everything I have in this work, my supervisor Ruth Gregory and Simon Ross, and also everyone else who, at one time or another, I spoke to and who gave me their advice and guidance.

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Chapter 1

Introduction

1.1 Geometry and Physics

Soap film surfaces provide a nice example of how a simple geometric principle can elegantly encompass a complicated and diverse physical phenomenon.

Geometrically a soap film is a minimal surface, or in other words, a surface which has vanishing extrinsic curvature [1]. Physically, on the other hand, the film formed on a wire frame dipped in a soap film solution has minimal area subject to the boundary constraints of the wire. Two circular loops of wire for example give us the surface shown below. In 1916 Einstein proposed his General theory of Relativity [6]

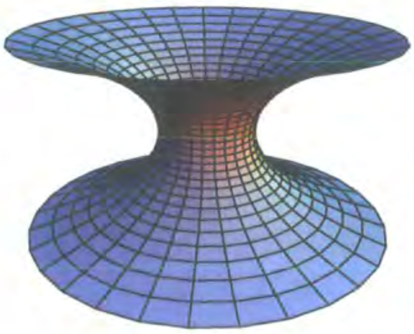

Figure 1.1: The Catenoid, whose vanishing mean curvature was discovered by Mesunier in 1778, is the only minimal surface of revolution.

in which gravity is a manifestation of the geometry of *space-time*. This is another, more profound, example of such a geometric principle.

1.2 String theory

General relativity as a theory modelling physics on large scales is very successful, it correctly yields for example the observed perihelion precession of Mercury, the observed bending of starlight by the sun and the red shift of distant objects [15]. There are however two clear (although not unrelated) issues it leaves to be addressed, the first is that it predicts the existence of black holes, the internal singularities of which imply the demise of the theory itself and the second is that it doesn't model small scale physics. Instead we need additional theories to describe short range phenomenon and Quantum Field Theory arose to meet this particular requirement, a framework which successfully unifies special relativity and quantum mechanics. The goal of unifying general relativity with quantum field theory underlies much of modern theoretical physics, the hope is to find a quantum theory of gravity describing all of nature.

The modern candidate for a quantum theory of gravity is *Superstring theory* [16] in which point particles are replaced as fundamental building blocks by strings with a length scale of $\sim 10^{-34}$m. Superstring theory, which for consistency lives in a space-time of ten dimensions, is in fact not unique. There are five consistent theories which appear to be related by various dualities, this in turn suggests that they themselves are different vacua of a more fundamental 11-dimensional theory known as M-theory [23]. The dualities led to the discovery that string theory admits higher-dimensional dynamical solitonic objects called *branes* which are non-perturbative in origin, these are surfaces in the space-time on which open strings can end, and have tensions proportional to the reciprocal of the string coupling. In the low energy limit they are the black p-branes[1] of supergravity [74] which are generalisations of black holes in various dimensions and as such are very interesting objects to study. As we have mentioned black holes in general relativity imply the demise of the

[1] A p-brane has a $(p+1)$-dimensional worldvolume.

theory itself, a problem we would hope string theory can resolve. An immediate problem however is how to deal with the additional dimensions that string theory seems to require. Historically T. Kaluza [8] attempted to unify the electromagnetic force with gravity by introducing an additional spatial dimension, his work was subsequently reformulated by O. Klein [9] which forms the basis of the modern point of view. Kaluza-Klein theory provided unification through the geometry of a higher-dimensional space-time, the modern use of their techniques is to systematically "remove" extra dimensions by supposing they are compact. The existence of gravitating higher-dimensional objects in string theory provides a framework on which we can model our observed four-dimensional universe, that is we confine ourselves to live on such an object. This leads us to the interesting possibility of an alternative to Kaluza-Klein compactification.

1.3 Braneworlds

The fundamental idea of a braneworld is that our universe is in fact an infinitesimally thin slice of a higher-dimensional space-time. All the fields of the Standard Model [19] are confined in some way to live on a brane, yet gravity, being the geometry of space-time itself, is allowed to propagate everywhere. A more abstract definition would be to define any model in which the gauge and gravitational interactions propagate on different spaces as a braneworld, particularly since the *relationship* between such interactions motivates the second part of this thesis. In principle of course the freedom the gravitational field would have in such a scenario would lead to inconsistencies in predictions with observational experience, this would be true for any higher-dimensional theory without a further mechanism to confine gravity in some way. In the previous section we mentioned briefly that Kaluza-Klein theory unified gravity and electromagnetism and the modern use of the techniques is to systematically remove additional dimensions in theories by supposing they are compact. The notion of a braneworld however provides us with a novel alternative.

Of particular interest are the Randall-Sundrum models [92, 93], the first, to be abbreviated to RS1, is introduced in detail in chapter 2. In this model we have two 3-branes of equal and opposite tension located at the fixed points of an $\mathbb{S}^1/\mathbb{Z}_2$ orbifold and separated by an anti-de Sitter bulk (i.e. a space-time with a negative cosmological constant). The orbifold construct seems a little arbitrary, however Horava and Witten [23] proposed that ten-dimensional heterotic string theory is related to an eleven-dimensional theory (M-theory) on a $\mathbb{R}^{10} \times \mathbb{S}^1/\mathbb{Z}_2$ space, they claim that if M-theory on such a space reduces to a string theory then it must be that of the heterotic string and as such has phenomenological applications. The orbifold construction is shown pictorially in fig 1.2. We find that the RS1 scenario provides a novel resolution of the *hierarchy problem*, namely the huge unexplained difference between the Planck and electroweak scales. Other attempts to explain this hierarchy typically result in the introduction of another large scale, such as the size of an extra dimension, and so are unsatisfactory. The RS1 model provides a solution *without* the need for large extra dimensions by introducing a *warp factor* in the metric which is a function of the extra coordinate.

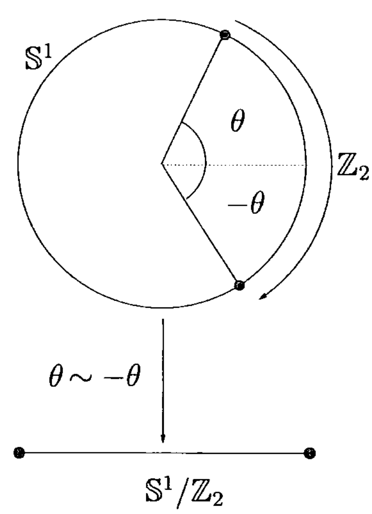

Figure 1.2: An orbifold construction. A discrete group acts on a space in some way and points in the same orbit are identified.

The second model, similarly abbreviated to RS2 and also described in detail chapter 2, doesn't provide us with a solution to the hierarchy problem but is far more interesting. Instead of two 3-branes located at orbifold fixed points a finite distance apart we have a single 3-brane with an *infinite* extra dimension, this latter fact is very unusual. Since gravity is free to propagate in all dimensions in these models any observer, brane based or not, would measure at the very crudest a five-dimensional version of Newton's law and so we would expect to rule out such a scenario immediately. However in RS2 the *warp factor* damps gravitational perturbations away from the brane and so perturbatively we still obtain an acceptable four-dimensional Newton's law for any brane based observer, this model therefore provides an *alternative* to compactification.

The braneworlds in the models of Randall and Sundrum have codimension 1, that is to say that they have a single direction transverse to the brane. In light of the possibility of the existence of extra dimensions motivated by string theory for example it's a natural question to ask if similar constructions are possible for braneworlds with *higher* codimension, a question which is not new [56]. The conclusion of the investigation in [56] is easily summarised: codimension 2 braneworld scenarios in Einstein gravity *can not* admit arbitrary brane based energy-momentum and so are inconsistent with experience, they could not yield realistic cosmological scenarios for example. Another way to look at this point is to observe that we are using worldvolume fields as an explicit source for bulk gravity and in order to recover the correct lower-dimensional gravity theory the geometry restricts the types of sources allowed. In codimension 1 there are also interesting modifications to the usual Friedman equations governing the cosmology that a braneworld observer would measure. This aspect of these models is also discussed in chapter 2 to more clearly highlight the codimension 2 result. In [53] we showed that the codimension 2 problem could be overcome if in addition to the usual Einstein tensor we included the Gauss-Bonnet terms in the equations of motion, which among other things such as being the leading order gravitational corrections in string theory ensure uniqueness of the gravitational equations of motion in more than four dimensions [4]. This analysis and a discussion of the results that we obtained is presented in chapter 3, specifically we show that with the Gauss-Bonnet terms the induced metric and energy-momentum tensors on the brane satisfy the usual four-dimensional Einstein equations. We then show that we can obtain the usual Friedman-Robertson-Walker cosmological scenario for a brane based observer even when carefully taking into consideration constraints imposed by the bulk equations of motion. The final sections of chapter 2 explain how the perturbation equations induced on a codimension 2 brane in Gauss-Bonnet gravity non-trivially have the correct four-dimensional tensor structure necessary in a realistic model.

1.4 Black holes and Thermodynamics

While black holes push the limits of what can be understood using general relativity, in four dimensions they are well understood; their horizons have topology $\mathbb{S}^2$, they are stable and are characterised by just their charge, their mass and their angular momentum [13]. Quantum mechanically however they are quite different objects; Hawking [45] showed that they can thermally radiate which together with the work of Bekenstein [59] and Bardeen, Carter and Hawking [48] led to a clear picture of black holes as being thermodynamic systems. That properties of the horizons of black holes, being rigorous results in differential geometry, can be identified with thermodynamical laws (such as entropy as summarised in the table below) which are based on microscopic statistical concepts is very surprising. Indeed understanding such a connection between the classical and quantum properties could shed more light on quantum gravity.

	Black holes	**Thermodynamics**
Zeroth Law	Surface gravity, κ, is constant over the horizon	T is constant throughout a body in equilibrium
1^{st} Law	$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ$	dE=TdS + work terms
2^{nd} Law	$\delta A \geq 0$	$\delta S \geq 0$
3^{rd} Law	Impossible to achieve $\kappa = 0$ through a physical process	Impossible to achieve $T = 0$ through a physical process

Table 1.1: Four laws of black hole mechanics.

As the temperature of a black hole solution is a quantum mechanical effect we ought to be better able to understand the relationship with our candidate for a quantum theory of gravity. To this end Strominger and Vafa [49] by explicitly counting microstates of extremal D-brane solutions found that their number agreed with the size of the entropy as calculated from the area of the event horizon. In fact the microstates, which we will assume are responsible for the thermodynamics, form a $(1+1)$-dimensional finite-temperature gauge theory living on the D-branes. This justification for the origin of the entropy must be seen as a success of string theory and motivated by it we can examine the deeper structure of the theory. Gubser and Mitra made a conjecture based on this and part of this thesis is devoted to it. First however we need to discuss other developments.

Further study of higher-dimensional black objects, such as the various p-branes of supergravity, led to an interesting discovery in complete contradistinction with black holes in four dimensions. Gregory and Laflamme showed [60, 61] that a class of p-brane solutions were classically *unstable*! That is they found that there existed a perturbation around a p-brane background which was spatially regular but grew exponentially in time, in fact they found that such an instability existed down to an extremal limit. Interestingly the p-branes they considered always had negative specific heat, that is they were thermodynamically unstable.

Given that certain black branes can be unstable we can ask the following interesting questions;

- What is the end state of an instability?
- Which other solutions are unstable?

In trying to justify the instability Gregory and Laflamme used a heuristic argument based on black hole thermodynamical laws as follows; consider a five-dimensional black string which is simply the product of a Schwarzschild solution with a line, $Sch_4 \times \mathbb{R}$. A portion of the string of length L and mass M has entropy proportional to M^2/L, a five-dimensional black hole on the other hand has entropy proportional to $M^{3/2}$. Thus for large L the black hole solution is entropically more favourable which indicates we might expect an instability while also suggesting what the end state of such an instability might be. Horowitz and Maeda [78] have shown however that a black sting solution can't have a collapsing $\mathbb{S}^1$ on its horizon and so the string can't dynamically pinch off as a result of the instability. This point is shown schematically in fig 1.3.

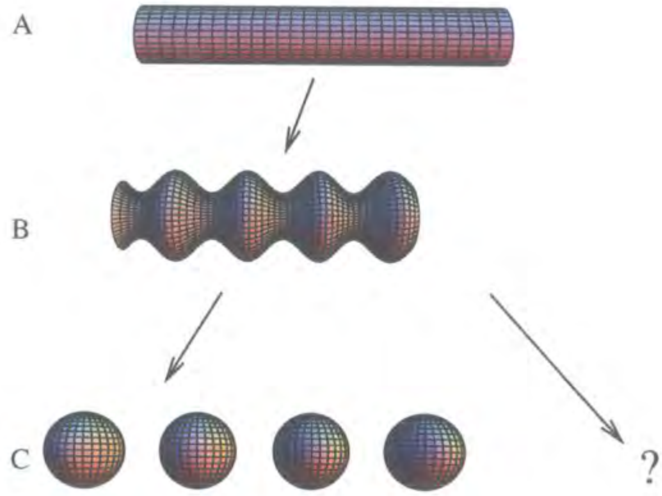

Figure 1.3: In the figure we have initially a solution (A) with an event horizon of topology $S^2 \times \mathbb{R}$, a small perturbation disturbs the solution as in (B) and the final state (C) is not possible through a classical evolution [78].

Motivated by the field theory interpretation of the microstates underlying p-brane thermodynamics and further by the AdS/CFT correspondence[2] Gubser and Mitra conjectured a precise relationship between thermodynamic stability and classical stability based on the following reasoning. A thermodynamic instability of a p-brane system could manifest itself in the dual finite-temperature gauge theory as a phase transition, this in turn should manifest itself in the existence of an exponentially growing mode which nucleates the new phase. Using the duality again such a mode in the field theory would correspond to an exponentially growing mode in real time, indicative of an instability. It is interesting to note here the connection between this proposal and the results in the literature regarding higher codimension braneworlds; in the former we see that the worldvolume field theory is proposed to restrict the geometry of the space-time through gravitational effects while in the latter the geometry of space-times admitting higher codimension braneworlds restricts the allowed worldvolume sources.

The Gubser-Mitra conjecture reads [62];

[2] A conjectured correspondence between superstring theory and gauge theory [50–52].

> "A black brane with a non compact translational symmetry is classically stable if and only if it is locally thermodynamically stable[3],"

and a partial proof was given for a certain class of p-branes by Reall [63]. We will give a review of his argument which is based on relating an unstable mode of spherically symmetric perturbations to a negative mode of a Euclidean black hole solution in chapter 4. Further investigations of the conjecture were also carried out in [68, 71–73]; the results up to that point supported the conjecture and a review of some of this evidence is also given in chapter 4. Studies of the connection between dynamical and thermodynamical instabilities which relax the requirement of translational invariance have also been performed and appear in [75–77].

In chapter 5 we extend the investigations of the conjecture to study smeared branes; that is, we take a p-brane and smear it uniformly over one of the transverse directions, and study stability to perturbations in this smeared direction. This is a natural extension of the investigation of p-branes in [63, 71–73]. The two classes of solutions are related by T-duality, which implies that the thermodynamics of the smeared branes is identical to that of the p-brane with the same total number of extended directions. However, the study of perturbations in the smeared direction is technically more challenging; certain simplifications exploited in [63] no longer apply as is explained in more detail in chapter 5. Specifically we exploited recent advances in the construction of *non-uniform* brane solutions [67], which were inspired by the development of Horowitz and Maeda mentioned previously. In particular we used an ansatz proposed by Harmark and Obers in [83] for such non-uniform solutions, which describes both vacuum black strings of the type discussed in [79, 80] and charged black branes smeared over a transverse circle. We show that smeared charged black holes provide a counter-example to the Gubser-Mitra conjecture, implying that the connection between dynamical instability and thermodynamics is more complicated than previously thought.

The thesis concludes in chapter 6 with a summary of our findings and suggestions

[3]Thermodynamic stability is taken to mean that the Hessian of the entropy (thought of as a function of extensive variables such as the charge and mass of the solution) has no positive eigenvalues.

for future research based on them, in addition there are two appendices giving more calculational details on results in the main text.

Chapter 2

Codimension 1 braneworlds

2.1 Introduction

This chapter gives a review of the success that the Randall-Sundrum braneworld models have had. First we give a brief discussion of the hierarchy problem and then how these models can provide a novel solution is explained in detail in section 2.2.1. As mentioned in chapter 1 an interesting aspect of braneworld models is the non-trivial interaction of the sources for fields on the brane with bulk gravity, in section 2.2.3 we explain how such non-linear gravity effects give rise to modifications to the standard cosmological Friedman equations that an observer on the brane would measure. This in particular sets the context for codimension 2 scenarios.

The final two sections then discuss how in the Randall-Sundrum scenario we can have on the one hand an infinite extra dimension and yet still recover conventional four-dimensional gravity contradicting conventional Kaluza-Klein wisdom, thus providing an *alternative* to compactification.

2.2 Randall-Sundrum I

There appear to be at least two fundamental energy scales in the universe we live in; the electroweak scale, $m_{EW} \sim 10^3$GeV and the Planck scale $m_{PL} \sim 10^{19}$GeV. The huge unexplained difference between the two is known as the *hierarchy problem.* There are some simple ways of trying to explain this hierarchy using theories with extra dimensions: suppose we have for example a $(4+n)-$dimensional factorisable space-time which is a product of a four-dimensional space-time with a flat n-dimensional compact space of volume V_n. Then if our higher-dimensional space-time has fundamental Planck scale M and the four-dimensional space-time has a fundamental Planck scale m_{PL} we find by performing the trivial integration over the extra dimensions that

$$m_{PL}^2 = M^{n+2} V_n. \tag{2.1}$$

Now the electroweak scale has been probed at distances $\sim m_{EW}^{-1}$ however gravity has not been probed anywhere near m_{PL}^{-1} and so the assumption that the latter is truly fundamental is based on the belief that gravity is not modified over ~ 30 orders of magnitude between where it is measured and the Planck length. Motivated by this simple argument it was proposed in [89–91] that the large hierarchy could be explained by making the extra dimensions very large with the doctrine that $M \sim m_{EW}$, so m_{EW} is the only fundamental short distance scale in nature. In such a scenario there would be no hierarchy among these scales, gravity is so weak as it is in some sense diluted by the large extra dimensions. This procedure however, while eliminating the present problem introduces another, namely the new hierarchy between the weak scale and the compactification scale, even for six extra dimensions (c.f. string theory) there is an unexplained difference in these scales of the order $\sim 10^5$ [35]. Randall and Sundrum proposed a space-time metric which was *not* factorisable, but in which there is an exponential factor multiplying the braneworld directions which is a function of the coordinate in the extra dimension, this exponential factor is responsible for generating the large hierarchy without introducing a new one. We will now present this in more detail.

2.2.1 The model

The RS1 scenario [92] consists of two 3-branes embedded in a five-dimensional space-time in which the bulk has possibly a cosmological constant Λ. The 3-branes are required to exhibit four-dimensional Poincaré invariance and so the metric is taken to be[1]

$$ds_5^2 = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu - r_c^2 d\phi^2, \tag{2.2}$$

where x^μ are the usual four-dimensional coordinates and $-\pi \leq \phi \leq \pi$ is a coordinate for the extra dimension, the size of which is determined by r_c. Since the space-time doesn't fill out all five dimensions we have to specify boundary conditions, these are taken to be periodicity in ϕ supplemented with the additional condition of identifying (x^μ, ϕ) with $(x^\mu, -\phi)$. That is to say we have imposed that the extra dimension has the topology of the orbifold space $\mathbb{S}^1/\mathbb{Z}_2$ (see fig (1.2) in chapter 1). The value of σ over the range of ϕ, taken to be $-\pi \leq \phi \leq \pi$, is then however completely specified when given on $0 \leq \phi \leq \pi$. We could of course rescale ϕ to eliminate r_c, this however would just change the periodicity condition and so we will leave it explicitly in the metric. The two 3-branes are now taken to reside at the orbifold fixed points, i.e. at $\phi = 0, \pi$ and form the boundary of the space-time. The orbifold singularities will provide, mathematically, the delta functions we need to support the branes. The metrics induced on the branes located at $\phi = 0, \pi$ are defined by

$$g_{\mu\nu}^{\phi=0,\pi} = G_{\mu\nu}(\phi = 0, \pi), \tag{2.3}$$

where $G_{\mu\nu}$ is the bulk metric defined in equation (2.2).
The classical gravity action which determines the physics can be split up as

$$\begin{aligned} S_{gravity} &= \int dx^4 \int_{-\pi}^{\pi} d\phi \sqrt{G}\left(-\Lambda + 2M^3 R\right), & (2.4)\\ S_{\phi=0} &= \int dx^4 \sqrt{g^{\phi=0}}\Lambda_1, & (2.5)\\ S_{\phi=\pi} &= \int dx^4 \sqrt{g^{\phi=\pi}}\Lambda_2. & (2.6) \end{aligned}$$

The details of any matter fields living on the branes is not important in this analysis, however we include possible cosmological terms to act as sources for the branes[2].

Einstein's equations following from (2.4)-(2.6) with the non-factorisable ansatz

[1] In this thesis we use a $\{+, -, -, -, \dots\}$ metric signature.
[2] A brane with no tension is the same as no brane as far as Einstein's equations are concerned.

(2.2) reduce to

$$\frac{6\sigma'^2}{r_c^2} = \frac{-\Lambda}{4M^3}, \tag{2.7}$$

$$\frac{3\sigma''}{r_c^2} = \frac{\Lambda_1}{4M^3 r_c}\delta(\phi) + \frac{\Lambda_2}{4M^3 r_c}\delta(\phi - \pi), \tag{2.8}$$

and a solution to the first equation with the orbifold symmetry is

$$\sigma = r_c|\phi|\sqrt{\frac{-\Lambda}{24M^3}}. \tag{2.9}$$

This makes the space-time between the two branes simply a slice of an AdS_5 geometry[3] as the solution only makes sense if $\Lambda < 0$. Differentiating equation (2.9) twice and using (2.8) gives us (see appendix (A.2) for how to differentiate functions with discontinuities and how to deal with delta functions in higher dimensions)

$$\Lambda_1 = -\Lambda_2 = 24M^3 k \qquad \Lambda = -24M^3 k^2, \tag{2.10}$$

where k, defined by these equations, is taken to be positive (we could choose k to be negative, however we can obtain this by redefining $\phi \to \pi - \phi$). Note also that this means there is necessarily a fine tuning between the brane tensions and the bulk cosmological constant (see section 2.2.3 for more on this point).

This completes the set up. We have two three-branes in a five-dimensional space-time located at the orbifold fixed points of $\mathbb{S}^1/\mathbb{Z}_2$ of the extra dimension, one brane has negative tension, the other has positive tension and they form the boundary of a slice of an AdS_5 space. Note that by taking into consideration the back reaction of the branes on the geometry we obtain a metric which depends on the position of the branes in the extra dimension, this dependence is exponential. Let us now examine the physical implications of this, in particular how the exponential dependence can resolve the hierarchy problem.

2.2.2 Hierarchy of scales

The first step to make is to identify the massless gravitational fluctuations, these provide the gravitational fields in the low energy *effective* theory as measured by an

[3] $AdS \equiv$ anti-de Sitter.

observer on one of the 3-branes. They take the following form

$$ds_5^2 = e^{-2kf(x)|\phi|}\bar{g}_{\mu\nu}dx^\mu dx^\nu - f^2(x)d\phi^2, \tag{2.11}$$

where $\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and k is defined in equation (2.10). So $h_{\mu\nu}$ is the fluctuation about Minkowski space and is the physical graviton of the four-dimensional effective theory. $f(x)$ measures the distance between the two branes and is referred to as the *radion* field. It can be shown that it doesn't affect the following calculation [33] so we will assume that it is frozen at the classical value $f(x) = r_c$, this is after all a toy model.

The four-dimensional effective theory is now obtained by substituting (2.11) in the action (2.4)-(2.6) and performing the integral over the extra-dimensional ϕ coordinate. It follows that

$$m_{PL}^2 = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right). \tag{2.12}$$

We should pause for a moment and compare this to the result for a factorisable space-time given in equation (2.1). Equation (2.12) tells us that the effective Planck scale depends only weakly on the five-dimensional scale if kr_c is large, it is also interesting to note at this point that even if $r_c \to \infty$, m_{PL}^2 still makes sense, neither of these facts are true for a factorisable space-time. In order to calculate how an observer on one of our 3-branes would measure the physical masses of fields we will consider the simple example of the Higgs field. Consider a fundamental Higgs field H bound to the brane located at $\phi = 0$, in this case the induced metric is just the usual Minkowski metric and the physics for matter living on this brane will have its usual form, for the brane located at $\phi = \pi$ this is no longer true as the argument of the exponential doesn't vanish there. In this latter case suppose we have a *five*-dimensional mass parameter m_0, then the matter part of the action on this brane is

$$S_{matter} = \int d^4x\sqrt{g_{\phi=\pi}}\left(g^{\mu\nu}_{\phi=\pi}\nabla^{(4)}_\mu H\nabla^{(4)}_\nu H - \lambda\left(|H|^2 - m_0^2\right)^2\right), \tag{2.13}$$

where all covariant derivatives are calculated with the induced metric on the brane. Using the solution given in section 2.2.1 we find

$$S_{matter} = \int d^4x\sqrt{\bar{g}}\left(\bar{g}^{\mu\nu}e^{-2kr_c\pi}\nabla^{(4)}_\mu H\nabla^{(4)}_\nu H - \lambda e^{-4kr_c\pi}\left(|H|^2 - m_0^2\right)^2\right), \tag{2.14}$$

where $e^{-2kr_c\pi}\bar{g}_{\mu\nu} = g_{\mu\nu}^{\phi=\pi}$. After rescaling the Higgs field using $H \to e^{-kr_c\pi}H$, so that the kinetic term is canonically normalised, we find

$$S_{matter} = \int d^4x\sqrt{\bar{g}}\left(\bar{g}^{\mu\nu}\nabla_\mu^{(4)}H\nabla_\nu^{(4)}H - \lambda\left(|H|^2 - \nu^2\right)^2\right), \tag{2.15}$$

where $\nu = e^{-kr_c\pi}m_0$ and is the scale by which the physical mass scales are set.

Recall that we are trying to resolve the hierarchy between the weak scale and the fundamental Planck scale, these differ in magnitude by a factor of $\sim 10^{15}$. If we propose that the five-dimensional Planck scale is *fundamental* then we can see from equation (2.12) that the effective four-dimensional Planck scale is of similar magnitude provided that $e^{-kr_c\pi}$ is small, moreover the five-dimensional fundamental mass scale is scaled by exactly this amount to give the analogous effective scale. Therefore if we have $kr_c \sim 50$ then the hierarchy goes away courtesy of the *exponential* scaling *without* introducing a new hierarchy. It is important to realise that this only occurs if *our* universe is identified with the 3-brane located at $\phi = \pi$, unfortunately the brane located at this point has negative tension and this is a problem as we will see in the end of the next section.

2.2.3 Cosmology

In this section we will examine cosmological models arising in the RS1 scenario. The mathematical formalism needed is discussed in detail in Appendix A.1, however let's briefly discuss the physics geometrically.

Einstein's equations govern how the geometry of space-time is affected by the energy in it. The geometry is described by the metric (it defines a notion of distance) and a *smooth* metric, via Einstein's equations, gives a smooth energy-momentum tensor. The delta function energy-momentum source of our braneworld is *not* smooth and so some derivatives of the metric must be at least discontinuous somewhere to support it, dealing with discontinuities like this in the highly non-linear context of general relativity is not trivial [106], however we can assure ourselves that the analysis makes sense geometrically. Perhaps the simplest most direct expression of this is to say that if we add singular energy-momentum to the space then "something must happen", in practice we introduce the singular behaviour artificially using assumptions such as $\mathbb{Z}_2$ symmetry[4]. The Gauss-Codazzi formalism provides a geometrical framework for describing how a submanifold curves in a higher-dimensional space, moreover it is able to do this in a covariant way. So we can convince ourselves that the analysis makes sense, the orbifold singularity creates the singular behaviour in the space-time for us and the Gauss-Codazzi formalism provides the technical tools. The whole analysis is brought together by using the Israel junction condition [22] which tells us how to treat general energy-momentum on the brane and will be discussed when we use it.

Consider an arbitrary time-like hypersurface $\mathcal{S}$ with a unit normal vector n_A embedded in a five-dimensional space-time, see fig 2.1 for example. Of course there are two possible choices for the normal, one for each side of the hypersurface, however we will not make a distinction here[5]. The induced metric and the extrinsic curvature of the hypersurface, discussed in appendix A.1, are defined by

$$h^M_{\ N} = \delta^M_N - n^M n_N, \tag{2.16}$$

$$K_{MN} = h^P_{\ M} h^Q_{\ N} \nabla_P n_Q. \tag{2.17}$$

To perform the required calculation we need three equations, two of them are collec-

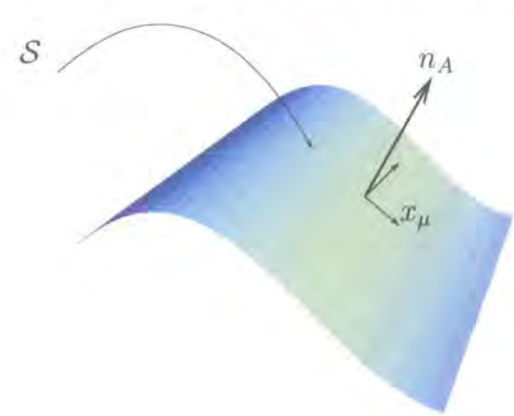

Figure 2.1: A codimension 1 hypersurface.

tively referred to as the Gauss-Codazzi equations and relate the extrinsic curvature

[4]Although also motivated by string theory, e.g. [23]

[5]In the end we will assume $\mathbb{Z}_2$ symmetry.

of the hypersurface to the intrinsic curvature of the five-dimensional space-time and the third is Einstein's equation. The Gauss equation is

$$R^{(4)}_{MNPQ} = h^R{}_M h^S{}_N h^T{}_P h^L{}_Q R_{RSTL} - 2K_{M[P}K_{Q]N}, \tag{2.18}$$

where $R^{(4)}_{MNPQ}$ is the four-dimensional Riemann tensor constructed using h_{MN} and the Codazzi equation is

$$\nabla^{(4)}_N K^N{}_M - \nabla^{(4)}_M K = n^P h^N{}_M R_{NP}, \tag{2.19}$$

where $\nabla^{(4)}$ is also computed with h_{MN}. Next we obtain the Ricci tensor from equation (2.18) by raising the first index and then contracting with the third as

$$R^{(4)}_{MN} = R_{PQ} h^P{}_M h^Q{}_N - R^P{}_{QRS} n_P n^R h^Q{}_M h^S{}_N + K K_{MN} - K^P{}_M K_{NP}. \tag{2.20}$$

We can then use this tensor to construct the Einstein tensor

$$\begin{aligned} G^{(4)}_{MN} &= \left(R_{PQ} - \frac{1}{2} g_{PQ} R\right) h^P{}_M h^Q{}_N + R_{PQ} n^P n^Q h_{MN} + \\ &+ K K_{MN} - K^P{}_M K_{NP} - \frac{1}{2}\left(K^2 - K^{PQ} K_{PQ}\right) h_{MN} - \bar{E}_{MN}, \end{aligned} \tag{2.21}$$

where

$$\bar{E}_{MN} = R^P{}_{QRS} n_P n^R h^Q{}_M h^S{}_N. \tag{2.22}$$

To introduce energy-momentum, embodied in the tensor T_{MN}, we need the five-dimensional Einstein equations

$$R_{MN} - \frac{1}{2} g_{MN} R = \kappa T_{MN}, \tag{2.23}$$

where $\kappa = \frac{1}{4M^3}$ to be consistent with notation in equation (2.4). Now the Riemann tensor in an arbitrary number of dimensions has more non-gauge degrees of freedom than are determined by Einstein's equations, these extra components are embodied in the Weyl-tensor[6] C_{MNPQ}. For example, gravitational waves are non-trivial solutions to Einstein's equations in vacuum [17]. The five-dimensional Riemann tensor can be decomposed in terms of this tensor in the following way [17]

$$R_{MNPQ} = \frac{2}{3}\left(g_{M[P} R_{Q]N} - g_{N[P} R_{Q]M}\right) - \frac{1}{6} R g_{M[P} g_{N]Q} + C_{MNPQ}. \tag{2.24}$$

[6]See page 29 for an explicit count of these components

If we substitute this decomposition into (2.21) and use the five-dimensional Einstein equations we find that

$$\begin{aligned} G^{(4)}_{MN} &= \frac{2\kappa}{3}\left(T_{PQ}h^P{}_M h^Q{}_N + \left(T_{PQ}n^P n^Q - \frac{1}{4}T\right)h_{MN}\right) + \\ &+ KK_{MN} - K^P{}_M K_{NP} - \frac{1}{2}\left(K^2 - K^{PQ}K_{PQ}\right)h_{MN} - E_{MN}, \quad (2.25) \end{aligned}$$

where

$$E_{MN} = C_{MPQN}n^P n^Q. \quad (2.26)$$

Up to this point we have been quite general, these equations hold for any time-like hypersurface in five-dimensional general relativity. Since we are interested in braneworld scenarios we will now be more specific. First, to be concrete, let's introduce a Gaussian normal coordinate system, specifically for each $p \in \mathcal{S}$ construct the unique geodesic through p with tangent vector n^M. Next choose an arbitrary coordinate system x^μ in a neighbourhood of p on $\mathcal{S}$ and label points in a neighbourhood of $\mathcal{S}$ by x^μ and a parameter ρ along the geodesic on which the point lies. What we

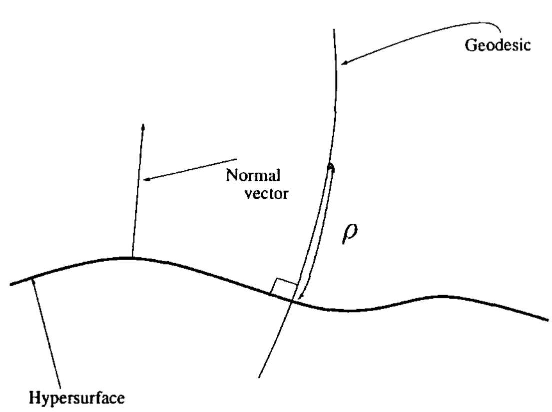

Figure 2.2: Gaussian normal coordinate construction.

have in fact done is to write the five-dimensional metric in a neighbourhood of the hypersurface as

$$ds_5^2 = h_{\mu\nu}dx^\mu dx^\nu - d\rho^2. \quad (2.27)$$

Now assign an energy-momentum tensor T_{MN} to the space-time

$$T_{MN} = -\Lambda g_{MN} + \hat{T}_{MN}\delta(\rho), \quad (2.28)$$

where from now on all braneworld objects will be hatted and in this case

$$\hat{T}_{MN} = -\Lambda_1 h_{MN} + \tau_{MN}. \tag{2.29}$$

So we have assumed that there is a bulk cosmological constant Λ and also (explicitly) a brane tension Λ_1 (or equivalently a cosmological constant on the brane), note that this latter split is ambiguous as we could soak up part of Λ_1 into the definition of τ_{MN}[7].

Next we need to make use of the following result

$$[K_{MN}] = \kappa\left(\hat{T}_{MN} - \frac{1}{3}g_{MN}\hat{T}\right), \tag{2.30}$$

where [...] is defined as

$$[f](x) = \lim_{\epsilon\to 0}\left(f(x+\epsilon) - f(x-\epsilon)\right). \tag{2.31}$$

It's known as Israel's Junction condition [22] and is very useful. Let's briefly explain how it arises in the Gaussian normal coordinate system. The metric in these coordinates is assumed to be continuous across the hypersurface, however an assumption of $\mathbb{Z}_2$ symmetry for example would mean that first derivatives w.r.t ρ would not be. The extrinsic curvature, containing at most only first derivatives of the metric would then be discontinuous at $\rho = 0$ and so normal derivatives of K_{MN} would then contain delta functions. Examining where such terms appear in (2.19) gives us the result (2.30) in this coordinate system. Using the general result (2.30) with the assumption of $\mathbb{Z}_2$ symmetry we find that

$$K^{\pm}_{MN} = \mp\frac{1}{2}\kappa\left(\hat{T}_{MN} - \frac{1}{3}g_{MN}\hat{T}\right), \tag{2.32}$$

where a + index means that the limit to zero is taken from above and a − from below. In other words with the assumption of $\mathbb{Z}_2$ symmetry the junction conditions completely determine the extrinsic curvature in terms of the brane energy-momentum.

So, with the assumption of $\mathbb{Z}_2$ symmetry and the choice of energy-momentum above we find that the four-dimensional Einstein tensor satisfies [37]

$$G^{(4)}_{MN} = 8\pi G_N \tau_{MN} - \Lambda_4 h_{MN} + \kappa^2 \pi_{MN} - E_{MN}, \tag{2.33}$$

[7] We can render this unambiguous in a cosmological context, see comments following equation (3.47) on page 42 in chapter 3.

with the tensor π_{MN} defined as

$$\pi_{MN} = \frac{1}{12}\tau\tau_{MN} - \frac{1}{4}\tau_{MP}\tau^{P}{}_{N} + \frac{1}{8}h_{MN}\tau_{PQ}\tau^{PQ} - \frac{1}{24}\tau^2 h_{MN}, \tag{2.34}$$

whereas Newton's constant G_N and the effective four-dimensional cosmological constant Λ_4 are defined via

$$8\pi G_N = \frac{\kappa^2}{6}\Lambda_1, \tag{2.35}$$

$$\Lambda_4 = \frac{1}{2}\kappa\left(\Lambda + \kappa\frac{1}{6}\Lambda_1^2\right). \tag{2.36}$$

There are now a few points to be made;

- To solve the hierarchy problem in section 2.2.2 we needed to assume that we lived on a brane of negative tension, we mentioned at the end of that section that there was a problem, we can now see one reason why. Equation (2.35) tells us immediately that if Λ_1 is negative the Newton constant would also be negative!

- In the RS1 scenario we found that the bulk cosmological constant was related to the brane tension in a precise way given by equations in (2.10). If we explicitly introduce the five dimensional Planck scale M through κ then we see that equation (2.36) implies that the four-dimensional effective cosmological constant Λ_4 has to be zero. This is another way of looking at the fine tuning referred to at the end of the section 2.2.1.

- The LHS of equation (2.33) is *not* Einstein's equation with the energy-momentum given in (2.29), we have modifications with possible observational consequences.

Let's look in more detail at what equation (2.33) implies in a purely cosmological context. To this end we will examine the equations for a homogeneous and isotropic braneworld described by the usual Friedman-Robertson-Walker metric [18]

$$\begin{aligned} ds_4^2 &= h_{\mu\nu}dx^\mu dx^\nu \\ &= d\tau^2 - a^2(\tau)d\mathbf{x}_c^2, \end{aligned} \tag{2.37}$$

where $d\mathbf{x}_c^2$ is the metric on a three-dimensional Euclidean space of constant curvature $c = 1, 0$ or -1 (sphere, plane or hyperboloid respectively).

In particular we assume that the braneworld energy-momentum is given by a homogeneous isotropic fluid of density $\rho(\tau)$ and pressure $p(\tau)$, in other words we have that

$$\hat{T}_{\mu\nu} = \rho\tau_\mu\tau_\nu + p\left(h_{\mu\nu} + \tau_\mu\tau_\nu\right), \tag{2.38}$$

where τ_μ are the components of the velocity vector of an observer. On top of these assumptions, to simplify things, we will also assume that E_{MN} is zero - E_{MN} is not directly related to the energy-momentum tensor, it is zero in the absence of purely gravitational excitations.

If we define, as usual, the Hubble parameter $H = \dot{a}/a$ where an overdot refers to differentiation with respect to τ, then equation (2.33) becomes [30]

$$H^2 = -\frac{c}{a^2} + \frac{8\pi G_N}{3}\rho + \left(\frac{\kappa}{6}\right)^2 \rho^2, \tag{2.39}$$

$$\dot{H} = \frac{c}{a^2} - 4\pi G_N(\rho + p) - 3\left(\frac{\kappa}{6}\right)^2 \rho(\rho + p). \tag{2.40}$$

These are not the usual Friedman-Robertson-Walker equations as they contain pieces quadratic in ρ and p. This means that cosmological evolution in this braneworld scenario is not the same as what we find in standard cosmology[8], an explicit example of how braneworld physics is affected by the higher-dimensional geometry.

2.3 Randall-Sundrum II

In this section we will review Randall and Sundrum's second paper [93]. The set up in this scenario is slightly modified from the previous one, if we look again at (2.12) we can see that the effective four-dimensional Planck scale is still well defined even if the periodicity of the extra dimension is taken to infinity, i.e. if the dimension is no longer *compact*. The scenario of a single brane in a space-time with an infinite extra dimension is exactly what we will discuss now.

Canonically with four non-compact spatial dimensions we would not even be able to reproduce the usual Newton law, there is simply too much room for gravity to move in. The RS2 scenario is able to confine gravity through the exponential warp

[8] Especially in the early universe when ρ was large.

factor. The analysis here to show this will follow that of Garriga and Tanaka [34]. Starting with the metric

$$ds_5^2 = e^{-2|y|/l}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{2.41}$$

where l is a constant related to the cosmological constant, defined in section 2.2.1. Similar to the procedure in section 2.2.2 we are going to consider a metric perturbation of the form

$$g_{MN} \to g_{MN} + h_{MN}. \tag{2.42}$$

Under a change of coordinates such as

$$x^M \to x^M + \eta^M(x^N), \tag{2.43}$$

the metric changes as

$$g_{MN} \to g_{MN} + \nabla_M \eta_N + \nabla_N \eta_M, \tag{2.44}$$

from which it is possible to show that one can choose the five functions η^M in such a way as to make

$$h_{5M} = 0. \tag{2.45}$$

The fifteen independent components of the perturbation tensor become ten, there is however still some residual gauge freedom as this doesn't fix the five functions η^M uniquely. The additional freedom can be used to ensure the location of the brane is fixed at $y = 0$, such a choice is known as the Gaussian normal gauge and here will be denoted by an overbar, i.e. $\bar{h}_{MN}$. Alternatively we can use the extra freedom to impose that both

$$h_{5M} = 0 \text{ and } h^\mu{}_\mu = h^\nu{}_{\mu,\nu} = 0, \tag{2.46}$$

which is known as the Randall-Sundrum gauge. In this latter case the location of the brane is not fixed but given by $y = -\eta^5$, however the linear order perturbation equations simplify nicely. In fact the perturbation equations are given by

$$\Delta_L h_{MN} = 0, \tag{2.47}$$

where Δ_L is the Lichnerowicz operator defined by

$$\begin{aligned}\Delta_L h_{MN} =& -\frac{1}{2}\Box h_{MN} - R_{MPNQ}h^{PQ} + R_{P(M}h_{N)}{}^P + \\ &+ \frac{1}{2}\nabla_M\left(\nabla_P h^P{}_N - \frac{1}{2}\nabla_N h\right) + \frac{1}{2}\nabla_N\left(\nabla_P h^P{}_M - \frac{1}{2}\nabla_M h\right),\end{aligned} \tag{2.48}$$

and in the Randall-Sundrum gauge they decouple and take the following form

$$\left(a^{-2}\Box^{(4)} + \partial_y^2 - 4l^{-2}\right) h_{\mu\nu} = 0, \tag{2.49}$$

where $a = \exp(-|y|/l)$ and $\Box^{(4)}$ is the four-dimensional Laplacian computed with the Minkowski metric.

The decoupling of the Lichnerowicz operator in the Randall-Sundrum gauge is very useful, we could for example remove the index structure if we wished which makes any analysis much simpler. The disadvantage is that the effect of the perturbation could be to alter the location of the brane, i.e. it would no longer be located at $y = 0$. We can get around this problem by using the Gaussian normal gauge and then transforming between the two. Since in both gauges the 55 and $\mu 5$ components vanish, any gauge transformation between them must satisfy

$$g^{\mu M}\nabla_M \eta_5 + \nabla_5 \eta^\mu = 0, \tag{2.50}$$

$$\nabla_5 \eta_5 = 0, \tag{2.51}$$

which respectively give the conditions

$$\eta^\mu = -\frac{l}{2} g^{\mu M}\partial_M \eta^5 + A^\mu(x^\rho), \tag{2.52}$$

$$\eta^5 = \eta^5(x^\mu). \tag{2.53}$$

It then follows that such a transformation between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ can be written as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - l\partial_\mu\partial_\nu\eta^5 + g_{\rho(\nu}A_{\mu)}{}^\rho - \frac{2}{l}g_{\mu\nu}\eta^5. \tag{2.54}$$

Now the junction condition on the extrinsic curvature in the Gaussian normal coordinate system is (c.f. equations (2.30) and (2.32))

$$\left(\partial_y + \frac{2}{l}\right)\bar{h}_{\mu\nu} = -\kappa\left(\hat{T}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\hat{T}\right), \tag{2.55}$$

where $g_{\mu\nu} = \exp(-2|y|/l)\eta_{\mu\nu}$. Here $\hat{T}_{\mu\nu}$ is *additional* energy-momentum on the wall, it doesn't for example include the contribution from the wall itself which we use to eliminate the background metric from the equation. Under our gauge transformation this last equation becomes

$$\left(\partial_y + \frac{2}{l}\right)h_{\mu\nu} = -\kappa\Sigma_{\mu\nu}, \tag{2.56}$$

where $\Sigma_{\mu\nu} = \left(\hat{T}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\hat{T}\right) + 2\kappa^{-1}\partial_\mu\partial_\nu\eta^5$. As a consequence of the symmetries of the space-time the bulk fields must be symmetric under the transformation $y \to -y$ and so the derivative at the origin has to be discontinuous[9]. Introducing delta functions to enforce these discontinuities in our perturbation equation, (2.49) then gives us

$$\left(a^{-2}\Box^{(4)} + \partial_y^2 - 4l^{-2} + 4l^{-1}\delta(y)\right) h_{\mu\nu} = -2\kappa\Sigma_{\mu\nu}\delta(y), \tag{2.57}$$

in the sense that integration of this equation over the wall reproduces equation (2.56). To solve this equation, even only formally, we need to know η^5. Since $h^\mu{}_\mu = 0$ we must have from equation (2.56) that $\Sigma^\mu{}_\mu = 0$, consequently

$$\Box^{(4)}\eta^5 = \frac{\kappa}{6}\hat{T}, \tag{2.58}$$

which can be thought of as an equation of motion for η^5. Of course we would need a specific energy-momentum tensor for matter on the brane to solve this. Since in this section we are trying to reproduce the usual Newton's law for matter on the brane we will be specific and choose

$$\hat{T}_{MN} = m_0\delta^{(3)}(\mathbf{x})\mathrm{diag}(1, \mathbf{0}, \mathbf{0}), \tag{2.59}$$

which is nothing more than the energy-momentum tensor for a single stationary point particle living at the origin of the spatial coordinates on the brane. Taking the trace of this energy-momentum and using equation (2.58) we find

$$\Box^{(4)}\eta^5 = \frac{\kappa m_0}{6}\delta^{(3)}(\mathbf{x}). \tag{2.60}$$

This last equation, assumed to be valid at $y = 0$, is simply Poisson's equation and has the well known solution (see for example [43] pg. 570)

$$\eta^5 = -\frac{\kappa m_0}{24\pi r}, \tag{2.61}$$

where $r = |\mathbf{x}|$. Now the formal solution to (2.57) can be written as

$$h_{\mu\nu} = -2\kappa \int d^4x' G_R(x, y; x', 0)\Sigma_{\mu\nu}(x'), \tag{2.62}$$

[9] As discussed on pages 16, 20 and 99

where the integration is over the $y = 0$ surface and G_R is a retarded Green's function. The RHS of this equation can be naturally split up into two parts using the definition of $\Sigma_{\mu\nu}$ as

$$-2\kappa \int d^4x' G_R(x, y; x', 0) \left(\hat{T}_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \hat{T}\right) - 4 \int d^4x' G_R(x, y, x', 0) \partial_\mu \partial_\nu \eta^5. \quad (2.63)$$

An appropriate choice of A^μ in equation (2.54) gives us

$$\bar{h}_{\mu\nu} = \frac{2}{l} \eta_{\mu\nu} \eta^5 - 2\kappa \int d^4x' G_R(x, y, x', 0) \left(\hat{T}_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \hat{T}\right). \quad (2.64)$$

Next we insert the explicit form of the energy-momentum tensor and the explicit form of η^5 to obtain

$$\bar{h}_{\mu\nu} = -\frac{\kappa m_0}{12 l \pi r} \eta_{\mu\nu} + \frac{2\kappa m_0}{3} \mathrm{diag}(2, 1, 1, 1) \int dt' G_R(x, y; x', 0). \quad (2.65)$$

It is possible to show that the retarded Green's function is given by [34] (a more detailed derivation is also presented in [36])

$$G_R(x, y; x', y') = -\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu (x^\mu - x'^\mu)} \left(\frac{a(y)^2 a(y')^2 l^{-1}}{\mathbf{k}^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \frac{u_m(y) u_m(y')}{m^2 + \mathbf{k}^2 - (\omega + i\epsilon)^2} \right), \quad (2.66)$$

where $k^\mu = (\omega + i\epsilon, \mathbf{k})$ and

$$u_m(y) = \frac{1}{N} \sqrt{\frac{ml}{2}} \left(J_1(ml) Y_2 \left(ml e^{|y|/l}\right) - Y_1(ml) J_2 \left(ml e^{|y|/l}\right)\right), \quad (2.67)$$

with N a normalisation constant satisfying $N = \sqrt{J_1(ml)^2 + Y_1(ml)^2}$ and with J_n and Y_n being Bessel functions of order n.

For the stationary case (which is the case for our point particle) it is more illustrative to consider the Green's function for the Laplacian operator given by

$$G(t, \mathbf{x}, y; \mathbf{x}', 0) = \int_{-\infty}^{\infty} dt' G_R(x, y; x', 0). \quad (2.68)$$

Again in [34, 36] it is shown that if both points are taken on the wall we have

$$G(\mathbf{x}, 0; \mathbf{x}', 0) = -\frac{1}{4\pi l r} \left(1 + \frac{l^2}{2r^2} + \ldots\right). \quad (2.69)$$

We can use this last equation to finally write

$$-2\kappa \int d^4x' G_R(x, y; x', 0) \left(\hat{T}_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \hat{T}\right) = \frac{\kappa m_0}{6 l \pi r} \mathrm{diag}(2, 1, 1, 1) \left(2 + \frac{1}{k^2 r^2} + O(1/r^3)\right), \quad (2.70)$$

so that (2.65) yields

$$\bar{h}_{\mu\nu} = -\frac{\kappa m_0}{4l\pi r}\left(\text{diag}(1,1,1,1) + \frac{1}{3k^2r^2}\text{diag}(2,1,1,1) + O(1/r^3)\right). \tag{2.71}$$

As this result is in the Gaussian normal gauge we have $\phi(r) = -\frac{1}{2}\bar{h}_{00}$ as the classical Newtonian potential and so we find without any further problems that

$$\phi(r) = \frac{m_0 G_5}{lr}\left(1 + \frac{2l^2}{3r^2} + O(1/r^3)\right), \tag{2.72}$$

where $\kappa = 8\pi G_5$ has been explicitly introduced to make the comparison with the familiar result more transparent. To which end if we further note that $G_4 = G_5/l$ (c.f. equation (2.12)) we have found the usual Newton potential for a point particle of mass m_0 with the addition of small corrections which *do not* contradict current experimental tests of Newton's inverse square law for gravitational attraction.

2.3.1 Graviton propagator

The appearance of the usual Newton's law for a stationary point mass in the previous section is at the least a necessary requirement of a good theory. However we know that the General Theory of Relativity provides a more accurate model of gravity and we should therefore also require that to some extent this stronger theory is reproduced in some way. To this end we can consider the structure of the massless graviton propagator. The matter part of the metric perturbation on the brane can be written as

$$h^{(m)}_{\mu\nu} = -2\kappa \int d^4x' G_R(x,0,x',0)\left(\hat{T}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\hat{T}\right) \tag{2.73}$$

and if we consider massless modes the retarded Green's function takes the following form (c.f. equation (2.66))

$$G_R(x,0,x',0) = -\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)}\left(\frac{l^{-1}}{k^2 - (\omega + i\epsilon)^2}\right). \tag{2.74}$$

If we insert the Green's function above into equation (2.73) we do not get the usual four-dimensional graviton propagator, this is because we find a factor of $\frac{1}{3}$ instead of the usual $\frac{1}{2}$. This difference is a direct result of being in five dimensions instead of four. However the full perturbation is given in equation (2.65) and includes a term proportional to η^5, in general we can formally solve the Poisson equation (2.58) for η^5 as

$$\eta^5 = \frac{\kappa}{6}\int d^4x' \frac{1}{\Box^4}\hat{T}, \tag{2.75}$$

where

$$\frac{1}{\Box^4} = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)}. \tag{2.76}$$

Putting it together we find that the full equation is,

$$h^{(m)}_{\mu\nu} = -2\kappa \int d^4x' \frac{1}{\Box^4}\left(\hat{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{T}\right). \tag{2.77}$$

We see the correct factor of $\frac{1}{2}$ is found so that the usual four-dimensional propagator appears as we would like. This is an important observation, it was shown in [31,32] that the difference in the tensor structures due to the different factors mentioned here is enough to give observationally inconsistent predictions for the bending of star light. To conclude this provides strong evidence that four-dimensional gravity is indeed reproduced.

Chapter 3

Codimension 2 models

3.1 Introduction

The geometry of a space-time exterior to a localised defect depends critically on the number of codimensions. Typically a localised defect of codimension 1 or 2 manifests its self-gravity in the global features of the space-time, this is not true for defects of codimension 3 and higher which we can see as follows; intrinsic spatial directions don't participate in the self-gravity interaction of the defect and so symmetry dictates that the self gravity will be manifested in the directions orthogonal to it, however in n-dimensions the Riemann and Ricci tensors have [37]

$$\frac{n^2(n^2-1)}{12} \quad \text{and} \quad \frac{n(n+1)}{2}, \tag{3.1}$$

independent components[1] respectively and so gravity in (1+1) and (2+1)-dimensions has no local degrees of freedom. The vacuum space-time exterior to a defect of codimension 1 or 2 is therefore locally flat and the gravitational effect of the defect shows up only globally. For codimension 1 scenarios we have two flat space-times glued across a boundary (i.e. an orbifold construct, see chapter 2) and for the codimension 2 string we have a conical space-time. The first situation has been well studied and important examples and aspects have been reviewed in chapter 2, it is the latter situation we will examine in this chapter.

[1]The difference between these is $\frac{1}{12}n(n+1)(n+2)(n-3)$ and counts the number of independent components of the Weyl tensor defined on page 19.

Before we do that, let's first discuss the results in the literature. Rubakov and Shaposnikov [58] tried to modify the canonical Kaluza-Klein ansatz with the intention of learning more about the cosmological constant problem, as a by-product they found that Einstein's equations for the six-dimensional metric

$$ds_6^2 = m(r)^2 ds_4^2 - dr^2 - L^2(r)d\theta^2, \tag{3.2}$$

where ds_4^2 is a maximally symmetric four-dimensional metric, admitted a very elegant analysis; Einstein's equations for the unknowns can be written as simple classical mechanical equations for particles in potential wells. Since codimension 2 models naturally have six dimensions this provides a very convenient framework for analysis. Using this framework Cline et. al. [56] found that in codimension 2 scenarios;

- The effect of adding a 3-brane is to introduce a conical singularity.
- Some solutions need space-time to be cut off with 4-branes.
- They observed that even with the more general metric

  $$ds_6^2 = N^2 dt^2 - M^2 d\mathbf{x}_3^2 - B^2 dr^2 - L^2 d\theta^2, \tag{3.3}$$

  where the functions can depend on t as well as r, the brane energy-momentum $\hat{T}_{\mu\nu}$ had to satisfy

  $$\hat{T}_{\mu\nu} \sim g_{\mu\nu}. \tag{3.4}$$

The last bullet point above is a real problem, it rules out any realistic codimension 2 models in Einstein gravity. Since it will turn out to be a special case of the main result of this chapter we will review it only in a simple case after a necessary review of conical singularities. In section 3.5 we present the work in the paper [53] where we showed that by considering Gauss-Bonnet terms this restriction on the energy-momentum tensor could be overcome, the rest of the chapter after that point then gives some important examples showing the features of the model.

3.2 Conical singularities

In this section we will see how the angular deficit of a conical space-time can be identified with the tension of a codimension 2 braneworld. To do this we will consider an explicit example [39]. Start with the following static cylindrically-symmetric metric

$$ds_6^2 = dt^2 - dz^2 - dr^2 - \beta(r)^2 d\phi^2, \tag{3.5}$$

where the coordinates are unrestricted except to say that, $r \in [0, \infty)$ and the periodic angular coordinate $\phi \in [0, 2\pi)$. The function $\beta(r)$ is given by

$$\beta(r) = \begin{cases} \frac{l}{\gamma}\sin\left(\frac{\gamma r}{l}\right) & (r \le l) \\ \left(r - l + \frac{l}{\gamma}\tan\gamma\right)\cos\gamma & (r > l), \end{cases} \tag{3.6}$$

and here $l > 0$ and $\gamma \in (0, \pi/2]$ are constants. In the interior region of the space-time $(r < l)$ there are two non-vanishing components of the Einstein tensor, specifically we have $G_{tt} = -G_{zz} = \gamma^2/l^2$ and hence an energy density

$$\rho = T^t{}_t = \gamma^2/l^2. \tag{3.7}$$

The metric in the exterior region $(r > l)$ has vanishing Riemann tensor.

Now the mass density of the cylinder μ, i.e. the density per unit length, is given by the integral of the energy density over the two-surface z, t=const. This gives

$$\begin{aligned} \mu &= \int_0^{2\pi}\int_0^l \frac{\gamma}{l}\sin\frac{\gamma r}{l} dr d\phi \\ &= 2\pi\left(1 - \cos\gamma\right). \end{aligned} \tag{3.8}$$

To model a string we then consider the limit $l \to 0$ (see fig. 3.1). In this case, since μ doesn't depend on l, the mass density converges trivially and we can tentatively assign

$$\mu_s = 2\pi\left(1 - \cos\gamma\right)\delta^{(2)}(r). \tag{3.9}$$

In the exterior region $\beta(r)$ can be put in the form of a standard metric for a conical space-time in four dimensions via the coordinate transformation $R = r - l + (l/\gamma)\tan\gamma$, specifically the metric would take the form

$$ds_6^2 = dt^2 - dz^2 - dR^2 - R^2\cos^2\gamma d\phi^2. \tag{3.10}$$

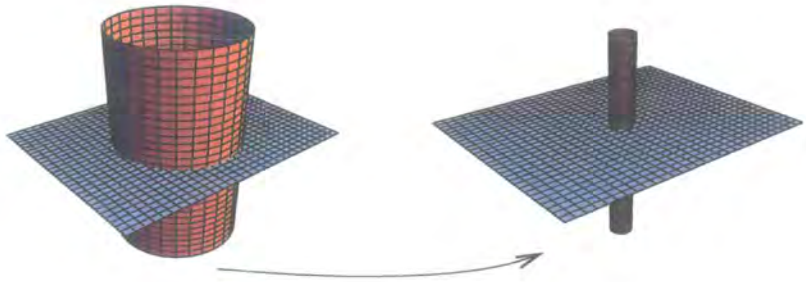

Figure 3.1: From a cylinder to a cone.

The angular variable ϕ, as already mentioned, has periodicity 2π, however an observer performing measurements around $R = 0$ would notice an angular deficit. This is because any observer measuring the radius and circumference of a circle located at $R = a$, say, would in fact measure the proper radius A and circumference C given by

$$C = 2\pi a \cos\gamma, \tag{3.11}$$

$$A = \int_0^a dR. \tag{3.12}$$

Now if our observer tries to relate these two measurements she will find an angular deficit Δ defined via

$$C = (2\pi - \Delta)\, A, \tag{3.13}$$

which using equations (3.11) and (3.12) can be seen to be equivalent to

$$\Delta = 2\pi\,(1 - \cos\gamma)\,. \tag{3.14}$$

This result is to be compared with (3.8) whence we see that the energy density of the string is exactly the angular deficit of the cone.

A string, being a localised defect of codimension 2, therefore gives rise to a conical geometry with an angular defect equal to the string's energy density and it's this string structure which will be our codimension 2 braneworld. Using Einstein's equations we can identify the singular structure through $\beta(r)$ as

$$\frac{\beta''}{\beta} = -2\pi[1 - \beta'(0)]\delta^{(2)}(r). \tag{3.15}$$

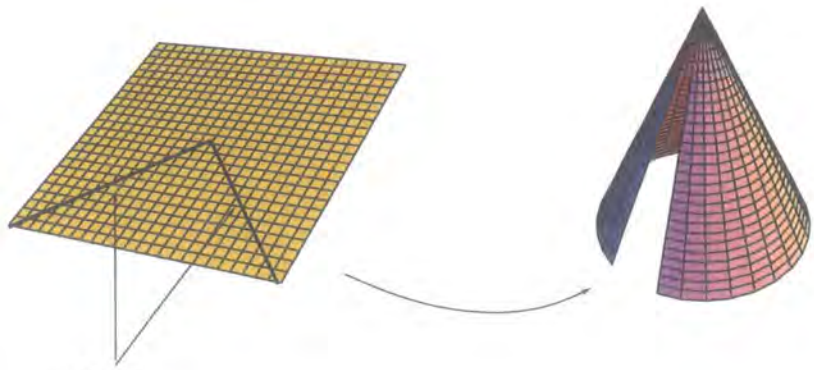

Figure 3.2: Conical space: How an angular defect gives a cone.

3.3 Explicit example

In this section we offer an explicit example of the third bullet point on page 30, that is we will examine the following special case of the metric given in equation (3.2) following Cline et. al. [56], namely

$$ds_6^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu - dr^2 - L^2 d\theta^2, \tag{3.16}$$

where all metric functions can depend on t as well as r.

We calculate Einstein's equations using this metric and then identify the terms which can support delta function sources. Assuming that the only metric field to vanish at $r = 0$ is $L(t, r)$ the terms we seek are those involving second derivatives of L w.r.t r as motivated in the previous section. It is straightforward to show for example that the Ricci tensor and Ricci scalar behave as

$$R_{\mu\nu} = 0 + \text{non-singular terms} \tag{3.17}$$

$$R = 2\frac{L''}{L} + \text{non-singular terms,} \tag{3.18}$$

and so any energy-momentum must satisfy

$$T_{\mu\nu} = -g_{\mu\nu}\frac{L''}{L} + \text{non-singular terms.} \tag{3.19}$$

Therefore the energy-momentum measured by a brane based observer must behave as $\sim g_{\mu\nu}$.

3.4 Gauss-Bonnet gravity

Einstein's equations are derivable from an action principle from which it follows in two dimensions there are no dynamics. In four dimensions this is, of course, not true. However there are terms which may be added to the action which are purely topological in four dimensions but which become important in five and higher-dimensional situations, the so called Gauss-Bonnet terms[2].

If we define the Einstein tensor G_{MN} by

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R, \tag{3.20}$$

then $G_{MN} + \Lambda g_{MN}$ is the most general combination of tensors in four dimensions which satisfies the following conditions [4];

- It is symmetric.
- It only depends on the metric and its first two derivatives.
- It has vanishing divergence.
- It is linear in the second derivatives of the metric[3].

It turns out that in five (and six) dimensions we can also satisfy these with a linear combination of the Einstein tensor, the metric and the *Lovelock* tensor [4, 5]. The required linear combination is obtained by varying the following action which is the usual Einstein piece plus quadratic terms proportional to α

$$S_{gravity} = S_{matter} + \int R + \alpha \left(R^2 - 4R_{MN}R^{MN} + R_{MNPQ}R^{MNPQ} \right). \tag{3.21}$$

In this way we see from a purely geometrical point of view that the theory described by the above action is a natural generalisation of the pure Einstein theory. This term can also be shown to arise from String Theory [28, 29] and since braneworlds themselves are certainly motivated by String Theory [23–25] we have yet more motivation that they should be considered.

[2]Generalisations of the Gauss-Bonnet theorem in higher even-dimensional spaces are possible via the Atiyah-Singer index theorem, the Euler character vanishes in spaces of odd dimensionality.

[3]In four dimensions this is implied by the other three.

3.5 The Model

The idea is now simple. We use the Gauss-Bonnet corrections in six dimensions to provide the additional structure we need to avoid the over restrictive energy-momentum requirement found by Cline et. al. in [56]. In [53] we proposed such a model which consists of a string-like defect - our codimension 2 braneworld, in a six-dimensional space-time in Einstein-Gauss-Bonnet gravity.

Our task is then to derive the effective equations on the brane (i.e. the codimension 2 equivalent of (2.33)). Before we do that it is worth comparing and contrasting with the codimension 1 scenarios. In codimension 1 there is a single normal vector to the braneworld, hence a single direction *from* the braneworld. This means first of all that there is a natural coordinate system adapted to the model, the *Gaussian normal* coordinates, essentially we choose the extra coordinate transverse to the brane as a measure of the proper distance along geodesics orthogonal to the braneworld (see fig (2.2) in section 2.2.3 of chapter 2). Secondly we can relate the intrinsic curvature of the space-time with the extrinsic curvature of the braneworld using the Gauss-Codazzi equations. In codimension 2 there are now *two* normal vectors and for a regular submanifold we can apply the Gauss-Codazzi formalism as before[4], however it is not possible to put two normal vectors at the location of the deficit with a well defined inner product as it depends on whether you measure the inner or outer angle. In this case we will therefore use a coordinate system defined in the vicinity of the braneworld and in which the effect of the braneworld itself appears formally as a delta function source, so instead of using the junction conditions to determine the extrinsic curvature we will identify the singular behaviour explicitly.

We will assume that our braneworld has a nonsingular metric, $\hat{g}_{\mu\nu}(x^\mu)$ and from now on all braneworld objects will be hatted. The coordinates x^μ denote braneworld directions and we will use Greek indices to refer to these. To construct the rest of the coordinate system consider the set of points which have topology $\mathbb{S}^1$ at fixed proper distance r from any x^μ and label them with the periodic coordinate θ which, without loss of generality, we will take to have periodicity 2π. The coordinates

[4] A discussion for arbitrary codimension is given in appendix A.1

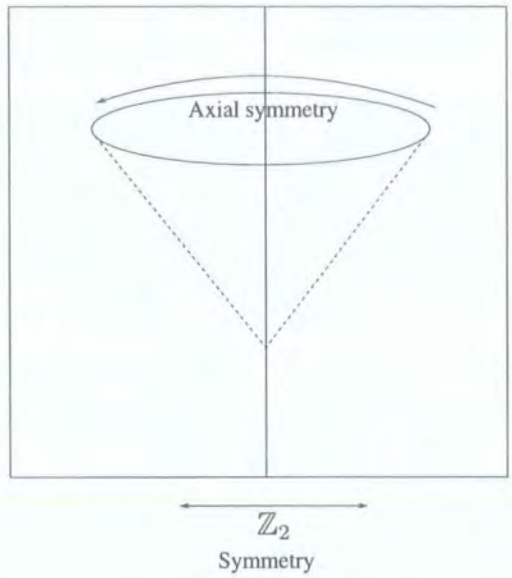

Figure 3.3: Assumption of axial symmetry.

x^μ, r and θ provide a full co-ordinatisation of the space-time in the vicinity of the braneworld which is located at $r = 0$.

There are still a few remaining assumptions we need to make. Firstly we will assume that the space-time is axially symmetric, we do this for two reasons;

- It simplifies the form of the metric in the bulk.
- It is a generalisation of the assumption of $\mathbb{Z}_2$ symmetry in the codimension 1 scenarios (see fig 3.3). It means for example that if the derivative w.r.t r of any field at the origin is not zero the derivative can't be continuous there.

Secondly there is of course some ambiguity in the labelling of θ, we will assume that this has been chosen so as to make the connection on the normal bundle vanish (equivalently the braneworld does not intersect itself, see appendix A.1 for more on this). The metric in these coordinates therefore takes the form

$$ds_6^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu - dr^2 - L^2(x^\mu, r) d\theta^2, \tag{3.22}$$

where the requirement of having a codimension 2 brane at the origin means that $L(x^\mu, 0) = 0$.

To obtain the braneworld equations of motion and hence some insight in to a description of the physics that an observer living on the brane would measure we now expand the metric in the neighbourhood of the braneworld since this is the region we are interested in. In particular with the assumptions we have so far

$$L(x^\mu, r) = \beta(x^\mu) r + O(r^2). \tag{3.23}$$

If we now substitute (3.23) into (3.22) and compare the result with the analysis in section 3.2 we find that if $\beta \neq 1$ we have a conical singularity at the origin which we interpret as being due to a delta function braneworld source. Strictly speaking, at least in Einstein gravity, we cannot talk of a delta-function source in terms of a zero-thickness limit of finite sources [106], to reiterate the discussion in chapter 2 the basic reason for this is that a smooth metric yields a smooth energy-momentum tensor and so the smoothness condition on the metric must fail, unfortunately the nonlinear nature of Einstein's equations make such a condition difficult to deal with rigorously. We however can avoid this technical issue and, geometrically, deduce the existence of a delta-function in the Riemann tensor from the holonomy of a parallely transported vector around the source. As the area of any loop vanishes the curvature has to become singular to account for the lack of parallel propagation, see fig 3.4. The equations of motion then make perfect sense with the delta-function

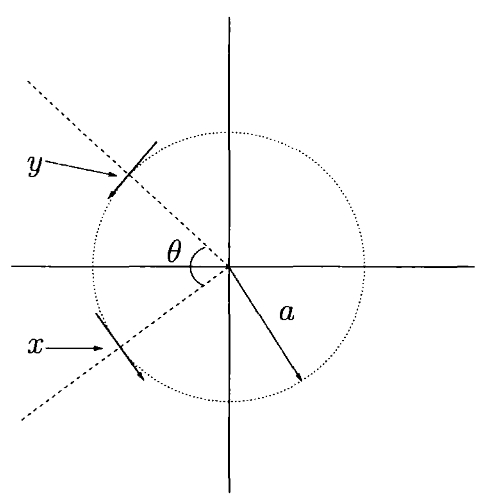

Figure 3.4: The dashed lines are identified, parallely transporting a vector from x around the circle to y induces a rotation irrespective of the size of a

being encoded in a notional discontinuity of the radial derivatives of the metric at $r = 0$. For a general braneworld then the problem we wish to solve is that of finding gravitating solutions that include the effect of a general brane energy-momentum tensor

$$T_{MN} = \begin{pmatrix} \hat{T}_{\mu\nu}(x)\frac{\delta(r)}{2\pi L} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.24}$$

where upper case Latin indices run over all the dimensions, the structure of the delta function is discussed in appendix A.2. In particular, we will be interested in the relation between the four-dimensional induced metric on the brane, $g_{\mu\nu}(x,0) = \hat{g}_{\mu\nu}(x)$, and the brane energy-momentum tensor, $\hat{T}_{\mu\nu}(x)$. It is this relation which determines the nature of the gravitational interactions that a "brane observer" would measure.

3.5.1 Analysis

To derive the equations satisfied by the induced metric our starting point is the Einstein-Gauss-Bonnet equation

$$M^4\left(G_{MN} + H_{MN}\right) = T_{MN} + S_{MN}, \tag{3.25}$$

where M is the six-dimensional Planck scale and

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R, \tag{3.26}$$

is the usual Einstein tensor and the Gauss-Bonnet contribution is given by

$$\begin{aligned} H_{MN} = \alpha\Big[& \tfrac{1}{2}g_{MN}(R^2 - 4R^{PQ}R_{PQ} + R^{PQST}R_{PQST}) \\ - & \; 2RR_{MN} + 4R_{MP}R_N{}^P + 4R^K{}_{MPN}R_K{}^P \\ & -2R_{MQSP}R_N{}^{QSP}\Big], \end{aligned} \tag{3.27}$$

with α a parameter with dimensions of $(mass)^{-2}$. S_{MN} is the bulk energy-momentum tensor, which we will not specify here, other than to assume that it has no delta-function contributions. If equation (3.25) is to be satisfied then there must be a singular contribution to the LHS with the structure $\sim \frac{\delta(r)}{L}$. As already discussed in section 3.2, such contributions arise as

$$\frac{L''}{L} = -(1-\beta)\frac{\delta(r)}{L} + \text{ (non} - \text{singular part)}. \tag{3.28}$$

Also we have

$$\frac{\partial_r^2 g_{\mu\nu}}{L} = \partial_r g_{\mu\nu} \frac{\delta(r)}{L} + \ (\text{non} - \text{singular part}). \tag{3.29}$$

In Einstein gravity these latter terms are zero as we will see later, however since they could in principle be nonzero here, we will retain them from now on. We must therefore set the delta-function contribution to the geometry equal to the brane energy-momentum tensor in order to solve the equations of motion. To simplify the calculation we note the following behaviour of some key pieces in the Einstein-Gauss-Bonnet equations

$$R^r{}_r = \frac{L''}{L} + (\text{non-singular part}), \tag{3.30}$$
$$R^\theta{}_\theta = \frac{L''}{L} + (\text{non-singular part}). \tag{3.31}$$

After some calculation, one obtains that the only singular part of the LHS of equation (3.25) lies in the μ, ν directions and is

$$-\frac{L''}{L}\left[g_{\mu\nu} + 4\alpha\left(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\right)\right] + \frac{\alpha}{L}\partial_r\left(L' \, W_{\mu\nu}\right), \tag{3.32}$$

where $W_{\mu\nu}$ is defined as the following combination of first derivatives of the four-dimensional metric

$$\begin{aligned} W_{\mu\nu} =& g^{\lambda\sigma}\partial_r g_{\mu\lambda}\partial_r g_{\nu\sigma} - g^{\lambda\sigma}\partial_r g_{\lambda\sigma}\partial_r g_{\mu\nu} + \\ &+ \frac{1}{2}g_{\mu\nu}\left[(g^{\lambda\sigma}\partial_r g_{\lambda\sigma})^2 - g^{\lambda\sigma}g^{\delta\rho}\partial_r g_{\lambda\delta}\partial_r g_{\sigma\rho}\right]. \end{aligned} \tag{3.33}$$

We can now use the properties

$$-\frac{L''}{L} = (1-\beta)\frac{\delta(r)}{L} + \ldots, \tag{3.34}$$
$$\frac{\partial_r(L' \, W_{\mu\nu})}{L} = \beta \, W_{\mu\nu}|_{r=0^+}\frac{\delta(r)}{L} + \ldots, \tag{3.35}$$

to obtain the matching condition by equating the $\frac{\delta(r)}{L}$ terms of equation (3.25). This yields

$$\boxed{2\pi(1-\beta)M^4\left[\hat{g}_{\mu\nu} + 4\alpha\hat{G}_{\mu\nu} + \alpha\frac{\beta}{1-\beta}\hat{W}_{\mu\nu}\right] = \hat{T}_{\mu\nu}} \tag{3.36}$$

where $\hat{G}_{\mu\nu}$ is the four-dimensional Einstein tensor for the induced metric, $\hat{g}_{\mu\nu}$, and $\hat{W}_{\mu\nu} \equiv W_{\mu\nu}|_{r=0^+}$. This is the main result of this chapter, the gravitational equations of a braneworld observer are the Einstein equations plus an extra Weyl term, $\hat{W}_{\mu\nu}$, which depends on the bulk solution. This term is reminiscent of the Weyl term in the codimension 1 braneworlds [37] (see also equation (2.33) in chapter 2), which gives rise to the corrections to the Einstein equations on the brane. Roughly speaking, the braneworld equation is obtained by taking the components of the full Einstein equations parallel to the brane, with the perpendicular components giving some information on the nature of the Weyl term. Depending on the symmetries present, in some cases (cosmology being the most physically interesting) we can completely determine the bulk metric, and hence these Weyl corrections. For codimension 2, the perpendicular components of the bulk equations do lead to constraints as we discuss presently, however these now no longer fix the bulk metric exactly, not even for the highly symmetric and special case of braneworld cosmology with Einstein gravity in the bulk. Let us now investigate the consequences of (3.36), including the consistency of the extra Weyl term, which arose as a result of allowing a discontinuity in the derivative of the parallel braneworld metric.

3.5.2 Discussion

A natural first check is to take the $\alpha \to 0$ limit to recover the Einstein case. Then equation (3.36) reduces to

$$2\pi(1-\beta)M^4\hat{g}_{\mu\nu} = \hat{T}_{\mu\nu}. \tag{3.37}$$

Although it looks like it is not possible to satisfy this condition unless the brane energy-momentum tensor is proportional to the induced metric, in fact we have not yet determined whether β is a constant. A non-constant β would correspond to a varying deficit angle, and is not determined by the braneworld equations alone. We must supplement the braneworld equations with the bulk equations normal to the braneworld, and since at the moment we wish to make as few assumptions as possible about the bulk, we will simply look at the divergent $\mathcal{O}(1/r)$ terms in the Einstein equations near the brane, as these cannot be cancelled by any regular bulk S_{MN}. These leading terms for the (μ,ν), (r,r) and (μ,r) components of Einstein's equations give

$$g_{\mu\nu}\frac{\langle L''\rangle}{L} - \frac{L'}{2L}\langle \partial_r g_{\mu\nu} - g_{\mu\nu}g^{\rho\sigma}\partial_r g_{\rho\sigma}\rangle = 0,$$
$$\frac{L'}{2L}g^{\rho\sigma}\partial_r g_{\rho\sigma} = 0, \quad \frac{\partial_\mu L'}{L} = 0, \tag{3.38}$$

where $\langle F\rangle$ stands here for the smooth part of F as we approach the brane. We now see directly that β must indeed be constant, and that

$$\partial_r^2 L|_{r=0^+} = 0, \tag{3.39}$$

$$\partial_r g_{\mu\nu}|_{r=0^+} = 0. \tag{3.40}$$

We can now confirm the observation of Cline et. al. [56], that Einstein codimension 2 braneworlds must have an energy-momentum proportional to their induced metric, and their gravitational effect is to produce a conical deficit in the bulk space-time, recovering two of the bullet points in the introduction to this chapter.

In Gauss-Bonnet gravity however the situation is not so simple since all these equations get corrections proportional to α and one cannot rule out the existence of solutions with $\hat{W}_{\mu\nu} \neq 0$. The $\mathcal{O}(1/r)$ terms in the (μ, r) components of the Einstein-Gauss-Bonnet equations for example are

$$-g^{\nu\sigma}\frac{\partial_\sigma L'}{L}\Big[g_{\mu\nu} + 4\alpha\Big(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\Big) - \alpha\, W_{\mu\nu}\Big] +$$
$$+ 2\alpha\frac{L'}{L}g^{\nu\sigma}\Big[\partial_r g_{\mu\nu}\, R^\rho{}_{\sigma\rho r} - \partial_r g_{\nu\sigma}\, R^\rho{}_{\mu\rho r}\Big] = 0, \tag{3.41}$$

with similar constraints from the $\mathcal{O}(1/r)$ terms of the (μ,ν) and (r,r) equations (though these are somewhat more complicated and not particularly illuminating). In this case we find that in general no simple restriction can be placed on the solution, and in particular the deficit angle β need no longer be constant.

However, it is important to note that some components of the Ricci curvature tensor (and scalar) are now divergent once we allow $\partial_r g_{\mu\nu}|_{0^+} \neq 0$. For example

$$R_{\mu\nu} = \frac{1}{2}\frac{L'}{L}\partial_r g_{\mu\nu} + \ldots = \frac{\partial_r g_{\mu\nu}}{2r} + \mathcal{O}(1), \tag{3.42}$$

near the brane. In a realistic situation, we could argue that a brane would have finite width, which could act as a cut-off for the curvature, hence all the results in this chapter would still be valid provided this cut-off is sufficiently large so that the curvature is still small compared to M^2, the six-dimensional Planck mass squared. In this smooth case, we *can* use the Gauss-Codazzi formalism and the θ-independence of the metric to write[5]

$$W_{\mu\nu} = K^{\lambda}_{i\mu} K_{i\nu\lambda} - K_{i} K_{i\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(K_{i}^{2} - K_{i\lambda\sigma}^{2} \right), \tag{3.43}$$

where $K_{i\mu\nu}$ are the two extrinsic curvatures ($i = 1, 2$) for each of the two normals. We therefore have the interpretation of $W_{\mu\nu}$ as being a geometric correction to the Einstein tensor due to the embedding of the braneworld in the bulk geometry. The interpretation is then that the Einstein equations acquire additional embedding terms which unfortunately cannot be deduced from the braneworld geometry alone.

The physical relevance of terms which lead to divergent curvatures and hence tidal forces in the vicinity of the braneworld is however questionable. If M is of order the (inverse) brane width, or if we wish to have a truly infinitesimal brane, then we are forced to conclude that for consistency we cannot stop at the Gauss-Bonnet curvature corrections, but must include all higher order curvature corrections thus entering a non-perturbative regime of which we can say nothing[6]. We are therefore led to impose $\partial_r g_{\mu\nu} = 0$, and equation (3.41) tells us that the deficit angle β is again constant and the equation for the induced metric (3.36) remarkably takes the form of purely four-dimensional Einstein gravity

$$\hat{G}_{\mu\nu} = \frac{1}{8\pi(1-\beta)\alpha M^4} \hat{T}_{\mu\nu} - \frac{1}{4\alpha} \hat{g}_{\mu\nu}. \tag{3.44}$$

We can read off our four-dimensional Planck mass as

$$m_{\mathrm{Pl}}^2 = 8\pi(1-\beta)\alpha M^4, \tag{3.45}$$

and we note the presence of an effective four-dimensional cosmological constant

$$\Lambda_4 = T_0 - 2\pi(1-\beta) M^4, \tag{3.46}$$

where T_0 is the bare brane tension

$$\hat{T}_{\mu\nu} = T_0 \ \hat{g}_{\mu\nu} + \delta T_{\mu\nu}. \tag{3.47}$$

[5]Compare to equation (2.25) on page 19.

[6]Of course if the curvature terms are in the Lovelock combinations [4] then they are purely topological.

Of course the splitting of the energy-momentum tensor in this manner is potentially arbitrary, however, for a cosmological brane $\delta T_{\mu\nu} \to 0$ as $t \to \infty$, and we can simply posit that $\delta T_{\mu\nu} \to 0$ as either t or $|\mathbf{x}| \to \infty$ as being a necessary requirement of a braneworld thus rendering (3.47) unambiguous.

Interestingly, the Einstein relation between β and the brane tension

$$T_0 = 2\pi(1-\beta)M^4, \tag{3.48}$$

no longer holds for Gauss-Bonnet gravity – we can specify the conical deficit and the brane tension independently, the only caveat being that if the Einstein relation does not hold, then we have an effective cosmological constant on the brane.

To sum up: we have found the equations governing the induced metric on the brane for a codimension 2 braneworld. We have shown that adding the Gauss-Bonnet term allows for a realistic gravity on an infinitesimally thin brane which remarkably turns out to be precisely four-dimensional Einstein gravity *independent* of the precise bulk structure, the only bulk dependence appearing via the constant deficit angle Δ in the definition of the four-dimensional Planck mass $m_{\mathrm{Pl}}^2 = 4\alpha\Delta M^4$. Since Einstein gravity appears quite generically, our model provides a novel alternative realisation of the infinite extra dimensions idea of Dvali et. al. [57]. Indeed, we could modify our model by adding braneworld Ricci terms (which can be motivated via finite width corrections to the brane effective action [104, 105]), which would give the same form of the braneworld gravity equations, and simply renormalize the four-dimensional Planck mass.

We also showed that it was possible to obtain a deviation from Einstein gravity via a non-zero $\hat{W}_{\mu\nu}$. In turn, this allows a variation of the bulk deficit angle and therefore the effective brane cosmological constant. In this case, one has to either perform a smooth regularisation of the brane by taking some finite width vortex model, or accept that the infinitesimally thin braneworld has a non-perturbative regime in the neighbourhood of the brane. Nevertheless it seems to be a very appealing feature toward a possible solution of the cosmological constant problem for example. One could envisage a situation in which the system is in a non-perturbative phase in which the cosmological constant can vary, and relax itself dynamically to a perturbative state in which the induced gravity on the brane is four-dimensional

Einstein gravity and with a very small cosmological constant (an infinite flat supersymmetric bulk might for instance lead to this situation [38]). Due to the unbounded curvature near the brane when this situation is violated it seems plausible that once the system reaches that configuration it would prefer to remain there.

3.6 Cosmology

In this section we will explicitly examine our model in a cosmological context, we have seen the general matching conditions that govern the geometry as measured by an observer on the brane, however as has been mentioned there are various constraints arising from the Einstein-Gauss-Bonnet equations in directions other than the four on the brane. In a cosmological context these are simple enough to deal with completely. First of all we will present a general inflating brane solution, this gives an example of a complete exact cosmological solution, and then we give a more general discussion of a Friedmann-Robertson-Walker scenario on a codimension 2 brane with an analysis of the restrictions imposed by the bulk equations of motion. A success of the codimension 1 models is their ability to essentially reproduce standard cosmology, here we see in detail the same success reproduced in codimension 2.

3.6.1 Inflating brane solutions

In order to obtain an inflating brane solution, we note that de Sitter space is simply a constant positive curvature space-time and as such is the analytic continuation of a four-sphere. We can write down six-dimensional Gauss-Bonnet-Schwarzschild solutions [54] as follows

$$ds_6^2 = V(r)dt^2 - \frac{dr^2}{V(r)} - r^2 d\mathrm{x}_4^2, \tag{3.49}$$

where

$$V(r) = \kappa^2 + \frac{r^2}{12\alpha}\left(1 \pm \sqrt{1 + \frac{12\alpha\Lambda}{5} + \frac{24\alpha\mu}{r^5}}\right), \tag{3.50}$$

and $d\mathrm{x}_4^2$ is a Euclidean four-space of constant curvature κ^2. This is a generalisation of the usual six-dimensional Schwarzschild solution in the sense that we obtain it if we take the *minus* sign in (3.50) and let $\alpha \to 0$. Note that doing the same

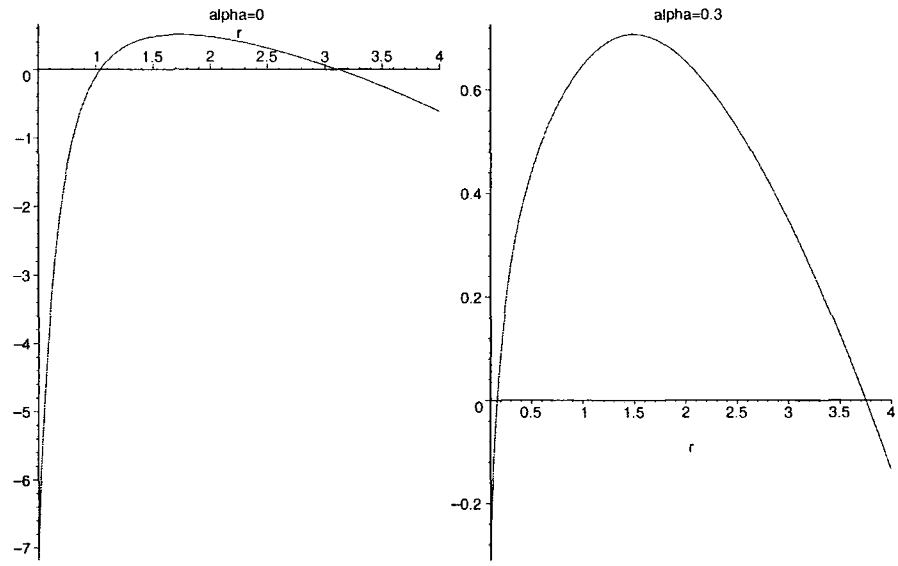

Figure 3.5: $V(r)$ plotted against r with $\kappa = \mu = \Lambda = 1$

thing with the opposite sign has no such analogue and the resulting metric is that for a space-time with a naked singularity. These two cases are often referred to as the Einstein and Gauss-Bonnet branches respectively, here we will work with the Einstein branch.

We doubly analytically continue the metric given in equation (3.49) as follows;

- First analytically continue the Euclidean four-space to a de Sitter space

$$d\mathbf{x}_4^2 \to d\tau^2 - e^{2\kappa\tau} d\mathbf{x}_3^2. \tag{3.51}$$

- Second analytically continue the t coordinate

$$t \to \imath\theta. \tag{3.52}$$

This gives us the following metric

$$ds_6^2 = r^2 \left(d\tau^2 - e^{2\kappa\tau} d\mathbf{x}_3^2\right) - \frac{dr^2}{V(r)} - V(r) d\theta^2. \tag{3.53}$$

Note that the periodicity of θ is not fixed as 2π but is determined by demanding the appropriate conical deficit at the analytically continued event horizon. To this end note that in general $V(r)$ has two real roots, see for example fig (3.5), and so we will calculate the angular defect at the location of each horizon and fix the periodicity of θ by smoothing one of the conical singularities out.

We can determine the angular deficit at the location of each horizon by expanding $V(r)$ around the horizon location, performing a change of variables and then doing a simple integral. Suppose the two zeros of $V(r)$ are located at $r_{1,2}$ (with $r_1 < r_2$) and that θ has periodicity η then the angular deficits $\Delta_{1,2}$ are;

$$\Delta_{1,2} = 2\pi \left(1 - \eta \left|\frac{V'(r_{1,2})}{4\pi}\right|\right). \tag{3.54}$$

The absence of a conical deficit at r_1 requires that $\Delta_1 = 0$, so we require

$$\eta = \left|\frac{4\pi}{V'(r_1)}\right|, \tag{3.55}$$

in which case

$$\Delta_2 = 2\pi \left(1 - \left|\frac{V'(r_2)}{V'(r_1)}\right|\right). \tag{3.56}$$

Alternatively, if instead we require that $\Delta_2 = 0$ then

$$\Delta_1 = 2\pi \left(1 - \left|\frac{V'(r_1)}{V'(r_2)}\right|\right). \tag{3.57}$$

In this latter case for generic $V(r)$ we would have an "angular excess" at r_1, for example in the situation depicted fig 3.5 (with $\alpha = 0.3$) we have,

$$\left|\frac{V'(r_1)}{V'(r_2)}\right| \approx 5.868, \tag{3.58}$$

so $\Delta_1 < 0$ and the interpretation would be an unphysical one as it would correspond to a negative tension object.

Motivated by this we smooth out the conical singularity located at $r = r_1$ which then determines the periodicity of θ as given by equation (3.55). This then gives us a space-time with a conical singularity, following the interpretation of the previous sections we have an exact codimension 2 brane world solution with an inflating cosmology.

3.6.2 FRW cosmology

Since we know that in our scenario Einstein's equations are reproduced for a brane based observer (although it is not yet clear that conventional gravity would be measured by such an observer as was motivated for codimension 1 scenarios in chapter 2) and, moreover, that the Friedmann-Robertson-Walker cosmological model is a good one we will examine these types of solutions and the constraints the bulk equations impose in our set up. So we start by taking the following form of the metric

$$ds_6^2 = N^2(t,r)dt^2 - M^2(t,r)\gamma_{ij}dx^i dx^j - dr^2 - L^2(t,r)d\theta^2, \tag{3.59}$$

where γ_{ij} is a maximally symmetric three-dimensional metric and lower case Latin indices will be understood to run over these three dimensions.

$$\gamma_{ij} = \left(1 + \frac{1}{4}\kappa\,\delta_{kl}\,x^k x^l\right)^{-2}\delta_{ij}, \tag{3.60}$$

with $\kappa = -1, 0, 1$ parameterising the spatial curvature. This is clearly a special case of equation (3.22). The coordinate θ is periodic with period 2π and the requirement of having a codimension 2 brane at the origin ($r = 0$) translates into $L(t,r)|_{r=0} = 0$. We then expand L thus defining $\beta(t)$ as in equation (3.23). Next we decompose the total energy-momentum tensor as the sum of two pieces, the brane and bulk terms. Note that in the analysis of section 3.5.1 we didn't make any assumptions about the nature of the bulk energy-momentum other than to assume it had no delta function sources, here of course the same restriction applies however we will assume that there is a bulk *cosmological constant.* So if we write

$$T_{MN} = T_{MN}^{brane} + T_{MN}^{bulk}, \tag{3.61}$$

then we are assuming that

$$T_{MN}^{brane} = \begin{pmatrix} -\rho N^2 \frac{\delta(r)}{L} & & \\ & pM^2\gamma_{ij}\frac{\delta(r)}{L} & \\ & & 0 \end{pmatrix}, \tag{3.62}$$

and for the bulk part, we will just assume that at $r = 0$ we can write it as

$$T_{MN}^{bulk}|_{r=0} = -\begin{pmatrix} g_{\mu\nu}\Lambda_x & \\ & g_{ab}\Lambda_r \end{pmatrix} \tag{3.63}$$

where the indices μ, ν run over t and x^i and a, b run over the extra-dimensional coordinates r, θ.

If we introduce, for convenience, the notation

$$M_0 = M(t,0), \tag{3.64}$$

then equation (3.36) becomes

$$(1-\beta)\big[1-12\alpha\frac{\dot{M}_0^2+\kappa}{M_0^2}\big]=\rho, \tag{3.65}$$

$$(1-\beta)\big[1-4\alpha\frac{\dot{M}_0^2+\kappa}{M_0^2}-8\alpha\frac{\ddot{M}_0}{M_0}\big]=-p. \tag{3.66}$$

It is clear from the equations above that when $\alpha = 0$ we can only find solutions if $p = -\rho$ (i.e. a cosmological constant equation of state), but importantly when $\alpha \neq 0$ we can find solutions that include matter on the brane and, moreover, this matter content determines a non-trivial cosmology. Also, from equations (3.65, 3.66) we can recover the conventional energy-momentum conservation law for matter on the brane, to do this we first note that the (t, r) component of the full Einstein-Gauss-Bonnet equations evaluated in the limit $r \longrightarrow 0$ gives

$$\lim_{r\to 0}\frac{\dot{L}'}{L}=\lim_{r\to 0}\frac{\dot{\beta}+\dot{\gamma}r+\mathcal{O}(r^2)}{\beta r+\mathcal{O}(r^2)}=0, \tag{3.67}$$

so $\dot{\beta}=\dot{\gamma}=0$. Now, differentiating equation (3.65) with respect to t and using again these matching equations we get the familiar result

$$\dot{\rho}=-3\frac{\dot{M}_0}{M_0}(\rho+p), \tag{3.68}$$

in fact performing the same procedure in the codimension 1 scenarios gives the same energy conservation result [30].

To continue the analysis we will consider the case where;

- $\rho = -p \equiv T$, so we assume a cosmological constant equation of state.
- $\kappa = 0$, zero spatial curvature.

In this case equations (3.65, 3.66) imply that the solution is simply dS space for the brane worldvolume ($\dot{M}_0^2/M_0^2 = \ddot{M}_0/M_0 \equiv H_c^2$ is constant) and the deficit angle can be determined from them as

$$1-\beta=\frac{T}{1-12\alpha H_c^2}. \tag{3.69}$$

However, these equations do not fix the value of H_c (and therefore the deficit angle), it will be determined by "bulk" physics, the main point of this section. To see this, consider the (r, r) component of the Einstein-Gauss-Bonnet equations for the metric given in equation (3.59), again evaluated at $r = 0$, that reads

$$\Lambda_r + \frac{3(\kappa + \dot{M}_0^2)}{M_0^2} + \frac{3\ddot{M}_0}{M_0} - 12\alpha\frac{(\kappa + \dot{M}_0^2)\ddot{M}_0}{M_0^3} + \\ - \frac{3M_2}{M_0}\left(1 - 4\alpha\frac{\kappa + \dot{M}_0^2}{M_0^2} - 8\alpha\frac{\ddot{M}_0}{M_0}\right) - N_2\left(1 - 12\alpha\frac{\kappa + \dot{M}_0^2}{M_0^2}\right) = 0, \tag{3.70}$$

where $M_2 = M''(t, 0)$ and $N_2 = N''(t, 0)$. This equation reduces for $\rho = -p$ and $\kappa = 0$ to

$$\Lambda_r + 6H_c^2 - 12\alpha H_c^4 - \left(\frac{3M_2}{M_0} + N_2\right)\left(1 - 12\alpha H_c^2\right) = 0. \tag{3.71}$$

From this equation we also see that $\left(\frac{3M_2}{M_0} + N_2\right) \equiv \Lambda_b$ has to be constant, and depends on the particular bulk solution we are considering. So H_c can be determined from the previous equation in terms of Λ_r, Λ_b and α.

This agrees, for $\alpha = 0$, with the general inflating solutions presented in the previous section for the Einstein case. But now, with $\alpha \neq 0$, we can consider the situation in which we have some matter on the brane. So let's take

$$\rho = T + \rho_m(t) \ , \ p = -T + \omega\rho_m(t), \tag{3.72}$$

with $\omega > -1$. In an expanding universe $\rho_{m(t\to\infty)} \to 0$, so asymptotically the solution must converge with the previous one. Then for $t \to \infty$ the matching conditions reduce to equation (3.69), and since β and T are time independent, we must fix the deficit angle to satisfy equation (3.69) at all times. Substituting back this value for β in equations (3.65, 3.66) we get

$$\frac{\dot{M}_0^2}{M_0^2} = H_c^2 - \frac{1 - 12\alpha H_c^2}{12\alpha T}\rho_m \tag{3.73}$$

$$\frac{\ddot{M}_0}{M_0} = H_c^2 + \frac{1 - 12\alpha H_c^2}{24\alpha T}(1 + 3\omega)\rho_m. \tag{3.74}$$

These are the conventional equations for the evolution of the scale factor of the universe with cosmological constant H_c^2 if we make the identification

$$\frac{1 - 12\alpha H_c^2}{12\alpha T} \to -\frac{8\pi}{3}G_N, \tag{3.75}$$

as can been seen by referring to equation (2.39). Notice that the effective cosmological constant is independent of the brane tension, since it is determined by the equation

$$\Lambda_r + 6H_c^2 - 12\alpha H_c^4 - \Lambda_b^{(\infty)}\left(1 - 12\alpha H_c^2\right) = 0, \tag{3.76}$$

where $\Lambda_b^{(\infty)} \equiv \lim_{t\to\infty}\left(\frac{3M_2}{M_0} + N_2\right)$ is a parameter to be determined by the bulk solution. Matter on the brane satisfying an equation of state such that $\omega > -1$ (so its energy density goes to zero as the universe expands) affects the evolution of the scale factor in a standard way.

3.7 Perturbation analysis

We know general relativity is in excellent agreement with physical observations, therefore we need to check that in our codimension 2 model an observer living on the brane would, at least to a first approximation, find this well established theory reproduced. In the codimension 1 scenario discussed in chapter 2 we were able to do this to some extent, to do this in the present case we will first perform a perturbation analysis (c.f. equation (2.49) in chapter 2).

To make the analysis as neat as possible we take the trace of the Einstein-Gauss-Bonnet equations and eliminate the Ricci scalar, this gives us a set of equations which look like "Ricci tensor + corrections" and will enable us to more easily identify the rôle played by the higher order curvature terms. In fact when we do this we find the equations (in n-dimensions) to be

$$\begin{aligned} R_{MN} - \alpha\Big[\frac{1}{2-n}g_{MN}(R^2 - 4R^{PQ}R_{PQ} + R^{PQST}R_{PQST}) + 2RR_{MN} - 4R_{MP}R_N{}^P + \\ - 4R^K{}_{MPN}R_K{}^P + 2R_{MQSP}R_N{}^{QSP}\Big] = T_{MN} + \frac{1}{2-n}g_{MN}T. \end{aligned} \tag{3.77}$$

The metric perturbation we are going to make is of the form

$$g_{MN} \longrightarrow g_{MN} + h_{MN}, \tag{3.78}$$

and we will also make a small additional perturbation to the components of the energy-momentum tensor

$$\delta T_{MN} \longrightarrow \delta T_{MN} + \overline{\delta T}_{MN}\delta^{(2)}(r), \tag{3.79}$$

with the assumption therefore that $\overline{\delta T}_{MN}$ is additional brane energy-momentum.

3.7.1 The equations

The details of the calculation of the equations of motion for h_{MN} to first order are given in appendix B.1. To be concrete we have considered a perturbation around a conical deficit in Minkowski space, i.e. a space-time with metric

$$ds_6^2 = dt^2 - dx^2 - dy^2 - dz^2 - dr^2 - L(r)^2 d\theta^2, \tag{3.80}$$

this is because locally any general metric in the codimension 2 scenario would take this form and therefore we expect it to highlight any generic features, moreover it is the simplest non-trivial case: quadratic corrections make no additional contributions to the equations in Minkowski space.

Working with the following notation

$$h = h^M{}_M, \quad h^4 = h^\mu{}_\mu, \quad h^\theta{}_\theta = \phi \quad \text{and} \quad h^\theta{}_\mu = A_\mu \tag{3.81}$$

the various components of the perturbation equations in Gaussian normal coordinates are;

- r, θ equation:

$$\text{Bulk:} \quad \partial_r \partial_\mu A^\mu = -\frac{1}{L}\partial_\theta \partial_r \left(\frac{h^4}{L}\right), \tag{3.82}$$

$$\text{Brane:} \quad \overline{\delta T}_{r\theta} = 0. \tag{3.83}$$

- μ, r equation:

$$\text{Bulk:} \quad \partial_r \left(\partial_\lambda h^\lambda_\mu - \partial_\mu h^4 + \partial_\theta A_\mu\right) = \frac{1}{L}\partial_\mu \partial_r (L\phi), \tag{3.84}$$

$$\text{Brane:} \quad \overline{\delta T}_{\mu r} = 0 \tag{3.85}$$

- μ, θ equation:

  Bulk:

$$-\frac{1}{2}\bar{\nabla}^2 A_\mu + \frac{1}{2}\partial_r^2 A_\mu + \frac{3}{2}\frac{L'}{L}\partial_r A_\mu + \frac{1}{2L^2}\left(-\partial_\theta \partial_\lambda h^\lambda_\mu + \partial_\theta \partial_\mu h^4\right) + \frac{1}{2}\partial_\mu \partial^\lambda A_\lambda = 0, \tag{3.86}$$

  Brane:

$$\overline{\delta T}_{\mu\theta} = 0. \tag{3.87}$$

- μ, ν equation:

  This equation is slightly more complicated, with

$$\delta R_{\mu\nu} = -\frac{1}{2}\bar{\nabla}^2 h_{\mu\nu} + \frac{1}{2}\partial_r^2 h_{\mu\nu} + \frac{1}{2L^2}\partial_\theta^2 h_{\mu\nu} + \frac{L'}{2L}\partial_r h_{\mu\nu} + \\ +\frac{1}{2}\left(\partial_\mu \partial_a h^a_\nu + \partial_\nu \partial_a h^a_\mu\right) - \frac{1}{2}\partial_\mu \partial_\nu h, \tag{3.88}$$

  we have the following equation

$$\delta R_{\mu\nu} - \alpha\frac{L''}{L}\left(g_{\mu\nu}(\bar{\nabla}^2 h^4 - \partial_\mu \partial^\lambda h^\mu_\lambda) - 2\bar{\nabla}^2 h_{\mu\nu} - 2\partial_\mu\partial_\nu h^4 + \right. \\ \left. +2(\partial_\mu \partial_\lambda h^\lambda_\nu + \partial_\nu \partial_\lambda h^\lambda_\mu)\right) = \overline{\delta T}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right). \tag{3.89}$$

- r, r equation:

  Here we find that the perturbation equation can be written as

$$\delta R_{rr} + \frac{1}{4}\alpha\delta(R^2 + ...) = -\frac{\Delta}{2}h^\theta_\theta \delta^{(2)}(r) + \overline{\delta T}_{rr} + \frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right), \tag{3.90}$$

  the LHS of which is more explicitly

$$\delta R_{rr} + \alpha\frac{L''}{L}(\frac{1}{4}(4g^{\lambda\rho}\delta R_{\lambda\rho} + 2\bar{\nabla}^2 h^\theta_\theta - 2\frac{L'}{L}\partial_r h^4 + \frac{4}{L^2}\partial_\theta \partial_\mu h^\mu_\theta - \frac{2}{L^2}\partial_\theta^2 h^4 - 2\partial_r^2 h^4)), \tag{3.91}$$

  where

$$\delta R_{rr} = -\frac{L'}{L}\partial_r h^\theta_\theta - \frac{1}{2}\partial_r^2 h. \tag{3.92}$$

  So simplifying as much as possible yields

$$-\frac{L'}{L}\partial_r\phi - \frac{1}{2}\partial_r^2 h^4 - \frac{1}{2}\partial_r^2\phi - \alpha\frac{L''}{L}(\bar{\nabla}^2 h^4 - \partial_\mu\partial^\lambda h^\mu_\lambda) = -\frac{\Delta}{2}\phi\delta^{(2)}(r) + \overline{\delta T}_{rr} + \\ +\frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right). \tag{3.93}$$

- θ, θ equation:

  Again we find (note how similar this is to (3.90))

$$\delta R_{\theta\theta} - \alpha\frac{1}{4}g_{\theta\theta}\delta\left(R^2 + ...\right) = g_{\theta\theta}\left(-\frac{\Delta}{2}h^\theta_\theta\delta^{(2)}(r) + g^{\theta\theta}\overline{\delta T}_{\theta\theta} + \right. \\ \left. -\frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right)\right), \tag{3.94}$$

where

$$\delta R_{\theta\theta} = -\frac{1}{2}\bar{\nabla}^2 h_{\theta\theta} - LL'\partial_r h^\theta_\theta - LL'' h^\theta_\theta - \frac{1}{2}L^2\partial_r^2 h^\theta_\theta + \partial_\theta\partial_\mu h^\mu_\theta - \frac{1}{2}\partial_\theta^2 h^4 - \frac{1}{2}LL'\partial_r h^4. \tag{3.95}$$

Putting it all together gives

$$\begin{aligned} &-\frac{1}{2}\bar{\nabla}^2 h^\theta_\theta + \frac{L'}{L}\partial_r h^\theta_\theta + \frac{L''}{L}h^\theta_\theta + \frac{1}{2}\partial_r^2 h^\theta_\theta + \partial_\theta\partial^\mu h^\theta_\mu + \frac{1}{2L^2}\partial_\theta^2 h^4 + \frac{L'}{2L}\partial_r h^4 + \\ &+\alpha\frac{L''}{L}(\bar{\nabla}^2 h^4 - \partial_\mu\partial^\lambda h^\mu_\lambda) = \frac{\Delta}{2}\phi\delta^{(2)}(r) + g^{\theta\theta}\overline{\delta T}_{\theta\theta} - \frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right). \end{aligned} \tag{3.96}$$

3.7.2 The analysis

As has already been used in section 3.5.1 the singular behaviour is embodied in the $\frac{L''}{L}$ term. In fact, to reiterate, we have the relationship

$$\frac{L''}{L} = -\Delta\delta^{(2)}(r), \tag{3.97}$$

where Δ is the angular deficit. As a boundary condition at the origin we match the singular behaviour in our equations. The r,r and θ,θ equations give us respectively

$$-\alpha\frac{L''}{L}(\bar{\nabla}^2 h^4 - \partial_\mu\partial^\lambda h^\mu_\lambda) = \left(-\frac{\Delta}{2}\phi + \overline{\delta T}_{rr} + \frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right)\right)\delta^{(2)}(r), \tag{3.98}$$

$$\begin{aligned} \frac{L''}{L}\left(\phi + \alpha(\bar{\nabla}^2 h^4 - \partial_\mu\partial^\lambda h^\mu_\lambda)\right) = &\left(-\frac{\Delta}{2}\phi + g^{\theta\theta}\overline{\delta T}_{\theta\theta} + \right. \\ &\left. -\frac{1}{4}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right)\right)\delta^{(2)}(r). \end{aligned} \tag{3.99}$$

Now by adding these last two equations and using equation (3.97) we obtain

$$\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta} = 0. \tag{3.100}$$

The μ,ν components similarly give

$$\begin{aligned} &-\alpha\frac{L''}{L}(g_{\mu\nu}(\bar{\nabla}^2 h^4 - \partial_\mu\partial^\lambda h^\mu_\lambda) + -2\bar{\nabla}^2 h_{\mu\nu} - 2\partial_\mu\partial_\nu h^4 + \\ &+2(\partial_\mu\partial_\lambda h^\lambda_\nu + \partial_\nu\partial_\lambda h^\lambda_\mu)) = \left(\overline{\delta T}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right)\right)\delta^{(2)}(r). \end{aligned} \tag{3.101}$$

Together equations (3.98), (3.100) and (3.101) then imply the following

$$\boxed{4\Delta\alpha\overline{\delta R}_{\mu\nu} = \overline{\delta T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{\delta T} + \frac{1}{2}g_{\mu\nu}\Delta\phi} \tag{3.102}$$

where $\overline{\delta R}_{\mu\nu}$ is the four-dimensional Lichnerowicz operator evaluated on a flat background without any gauge choice, we see that the correct factor of $\frac{1}{2}$ is found even though we are in a six-dimensional space-time as was also found in the analysis of the Randall Sundrum scenario described in section 2.3.1 of chapter 2. We can also see that the addition of $\overline{\delta T}_{rr}$ and $\overline{\delta T}_{\theta\theta}$ terms doesn't affect the final equations governing the braneworld physics. The equations are, however, not quite the usual Einstein equations, we would also need ϕ to be zero.

The condition that $\phi = 0$ on the brane is equivalent to the angular deficit and hence the brane tension remaining constant in the Gaussian normal gauge, this can be seen as follows; if the proper radius and circumference of a circle around the origin are denoted by R and C respectively then

$$R = \int_0^{\epsilon} \sqrt{|g_{rr}|}dr, \tag{3.103}$$

$$C = \int_0^{2\pi} \sqrt{|g_{\theta\theta}|}d\theta, \tag{3.104}$$

so under a general metric perturbation we would find that to first order

$$R \sim \epsilon\left(1 - \frac{1}{2}h_{rr}\right), \tag{3.105}$$

$$C \sim 2\pi L(\epsilon)\left(1 + \frac{1}{2}h^{\theta}_{\theta}\right). \tag{3.106}$$

In particular in the chosen gauge we have that

$$\frac{C}{R} \sim 2\pi\frac{L(\epsilon)}{\epsilon}\left(1 + \frac{1}{2}\phi\right), \tag{3.107}$$

and if we take the limit as $\epsilon \to 0$ the angular deficit after the perturbation $\overline{\Delta}$ can be seen to satisfy

$$\overline{\Delta} = \Delta - \pi L'(0)\phi, \tag{3.108}$$

which in the notation of this section implies

$$\delta\Delta = -\pi L'(0)\phi, \tag{3.109}$$

and hence the result as promised.

3.8 Summary

In this chapter we have shown how the problems encountered by codimension 2 braneworlds in Einstein gravity can be overcome with the introduction of Gauss-Bonnet terms, in particular the braneworld equations of motion are Einstein's equations with additional Weyl terms as discussed earlier. We have shown via a general calculation and a specific example that the resulting braneworld physics is at least as rich as the usual Einstein theory, importantly we find that we can reproduce the usual FRW cosmology in simple cases thus reproducing a success of the codimension 1 models.

For the codimension 1 Randall-Sundrum scenario it was possible to show that to a first approximation the usual Newton law would be found by an observer on the brane, an analogue of this argument is so far lacking in our codimension 2 Gauss-Bonnet scenario. Having said that we are able to show that for a braneworld observer the correct four-dimensional tensor structure for perturbations is found even though the space-time is fundamentally six-dimensional, moreover as the equations of motion for the braneworld perturbations don't depend on the radial or transverse components we suggest that reproducing Einstein gravity on the brane is not a result of zero modes being confined there but rather an exclusion principle.

Chapter 4

Stability and Thermodynamics

4.1 Introduction

This chapter and the next will be concerned with a possible relationship between dynamical and thermodynamical instabilities. In the first part of this thesis we have seen how worldvolume fields are restricted by bulk gravity, now motivated by the existence of a finite temperature field theory dual to black p-brane solutions Gubser and Mitra conjectured a precise relationship between these two types of instabilities [62]. That is, they proposed how the worldvolume field theory description of the microstates underlying black hole thermodynamics affects the geometry of a solution through gravitational effects.

Classically a black hole is a region of space-time where gravity has become so strong that nothing, not even electromagnetic radiation, can escape. An important development in the study of black hole physics was Hawking's discovery [46] that quantum mechanically they could thermally radiate with temperature

$$T = \frac{\kappa}{2\pi}, \tag{4.1}$$

where κ is the surface gravity. This development suggested that the laws of black hole mechanics (summarised in table 1.1 on page 6) should not only be analogous to thermodynamic laws but *are* thermodynamical in nature. Hawking radiation is a purely quantum phenomenon, this suggests that a candidate quantum theory of gravity should be able to help us better understand the origin of these thermodynamic properties which in turn should help us to better understand quantum gravity. Another important development in black hole physics was the discovery of Gregory and Laflamme [60,61] that in higher dimensions black hole solutions can be dynamically unstable. In fact the study of black holes in five, six or higher dimensions is much richer than in four. While in four dimensions the static neutral black hole is given by the Schwarzschild solution in higher dimensions we can consider other interesting black hole solutions. The study of solutions with horizons with non-trivial fundamental groups such as $\mathbb{R}^{n-1} \times \mathbb{S}^1$ are potentially more complicated and richer, on the one hand because a black hole on a cylinder has self interactions, easily seen by passing to the universal cover[1], where we end up with an array of black holes and non linear interactions between them give the complications, on the other hand the radius of the circle provides another scale in the problem giving a richer structure. In the simplest case, a Schwarzschild solution smeared over an infinite transverse direction was shown by Gregory and Laflamme to be dynamically unstable to perturbations with wavelength approximately seven times the Schwarzschild radius [60]. This is in complete contradistinction with the fact that in four dimensions black holes are dynamically stable. A review of the existence of this instability is given in section 4.2. The rest of this chapter will then be concerned with the conjecture of Gubser and Mitra and motivation for it, sections 4.3.1 and 4.4 contain a partial proof and various motivating examples.

4.2 Gregory-Laflamme instability

As mentioned in the previous section there is a substantial difference between black holes in four dimensions and those in higher dimensions. For example Hawking showed that the event horizon of a black hole in four dimensions is necessarily topologically a sphere [44], he argues that a space-like slice of the horizon is a connected, orientable 2-surface which admits a metric of positive scalar curvature and hence from the Gauss-Bonnet theorem must have $\chi = 2$ from the well known classification

[1]The universal cover of a connected topological space X is a unique simply connected space which has the same local properties as X. See for example [70]

of compact, oriented 2-surfaces [70]. However even in five dimensions things are very different, the techniques he applied do not generalise (the Euler character appearing in the Gauss-Bonnet theorem is identically zero for odd-dimensional spaces) and moreover it is easy to write down five-dimensional solutions with event horizons that have topology $\mathbb{S}^3$ or $\mathbb{R} \times \mathbb{S}^2$ for example.

It is well known that the topologically spherical solutions are *stable*, that is, small perturbations do not grow unbounded in time [13]. However for the black string, under certain circumstances, this is not true as was shown by Gregory and Laflamme. It is this instability that we shall review in this section.

4.2.1 Perturbation ansatz

The discussion here is based on the original work of Gregory and Laflamme [60]. They showed that the following ten-dimensional black string solution

$$ds_{10}^2 = ds_{Schwarzschild}^2 - dx_i dx^i, \tag{4.2}$$

where $ds^2_{Schwarzschild}$ is the usual D-dimensional Schwarzschild solution and lower case Latin indices run from 1 to $10 - D$, admits a dynamical instability. In order to show this an analysis of the linear order perturbation equations is required and to this end we will consider a metric perturbation of the form

$$g_{MN} \to g_{MN} + h_{MN}, \tag{4.3}$$

and work in the transverse trace free gauge, i.e. where $h^M_{\ M} = 0$ and $h^M_{\ N;M} = 0$. Now to show that the solution is unstable it is sufficient to find any instability, it turns out to be enough to consider just a simple s-wave (physically we would expect higher angular momentum modes to be more stable) of the form

$$h^{\mu i} = 0, \tag{4.4}$$

$$h^{ij} = 0, \tag{4.5}$$

$$h^{\mu\nu} = e^{\Omega t} e^{i\mu_i x^i} \begin{pmatrix} H^{tt}(r) & H^{tr}(r) & 0 & 0 & \dots \\ H^{tr}(r) & H^{rr}(r) & 0 & 0 & \dots \\ 0 & 0 & K(r) & 0 & \dots \\ 0 & 0 & 0 & \frac{K(r)}{\sin^2\theta} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.6}$$

where r is the usual spherical polar radial coordinate in D-dimensions, as for certain values of Ω and $\sum \mu_i^2$ solutions to the Lichnerowicz equations exist.

Note that when dealing with problems like this it, in principle, may be very difficult to establish whether any solutions correspond to real physical instabilities or are simply artifacts of the coordinate system which we may have chosen to work with. In the present case however there is a simple way to address this issue. Using the metric (4.2) and the s-wave assumption for the perturbation we can write the ten-dimensional Lichnerowicz equation in the following way

$$\left(\Delta_L^D + \mu^2\right) H_{\mu\nu} = 0, \tag{4.7}$$

where Δ_L^D is the D-dimensional Lichnerowicz operator and $\mu^2 = \sum_i \mu_i^2$. Now since a *pure* D-dimensional gauge perturbation satisfies the Lichnerowicz equation for a vacuum space-time it follows that *if* $\sum_i \mu_i^2 \neq 0$ then the perturbation *must* be physical.

4.2.2 Asymptotic and near horizon geometries

Finding an exact solution to (4.7) is too hard. Instead what Gregory and Laflamme did was to find the form of the solutions both asymptotically and near the horizon, identify which were regular and then numerically integrate in a regular solution from infinity and see if it could be matched with a regular near horizon solution. Although this procedure depends crucially on the form of the perturbation equations we can calculate the asymptotic behaviour quite easily, we also note the near horizon behaviour which depends on the Hawking temperature.

- **Asymptotic behaviour**

  Assuming that the space-time is asymptotically flat then asymptotically the Lichnerowicz equation can be written, in the DeDonder gauge, as

$$\bar{\nabla}^2 h^{\infty}_{MN} = 0, \tag{4.8}$$

  where an overbar refers to the operator in flat space. Using the s-wave form of the metric perturbation we can solve this equation to give

$$h^{\infty}_{..} = A_+ e^{\sqrt{\Omega^2+\mu_i^2}r} + A_- e^{-\sqrt{\Omega^2+\mu_i^2}r}, \tag{4.9}$$

  where, of course, we require $A_+ = 0$ for a regular solution.

- **Near Horizon behaviour**

  For this particular example the Lichnerowicz equations can be reduced to a single differential equation for the unknown H^{tr} field. Its near horizon behaviour can be written as

$$H^{tr}(r) = B_+(r - r_+)^{-1+\frac{\Omega}{4\pi T_H}} + B_-(r - r_+)^{-1-\frac{\Omega}{4\pi T_H}}, \tag{4.10}$$

  where r_+ is the horizon radius and T_H is the Hawking temperature of the solution.

4.2.3 Existence of solutions

For the specific case of the black string we have

$$T_H = \frac{D-3}{4\pi r_+}, \tag{4.11}$$

where $r = r_+$ is the horizon location. For the solution to be regular at the horizon we need $B_- = 0$ and $\Omega > 0$ in equation (4.10). It turns out that for $\Omega > (D-3)/r_+$ we can rule out the existence of instabilities of this type analytically. If this inequality is not satisfied then, as we have already said, we have to resort to numerical techniques. The results of this numerical investigation are presented in [60], they found that there are indeed solutions to (4.7) that are regular on the horizon and decay at spatial infinity, moreover there is a critical value μ_* of μ such that for $\mu < \mu_*$ there is always such a solution. For $r_+ = 2$ we find for various D that,

D	μ_*
4	0.44
5	0.63
6	0.92

Table 4.1: Threshold values

Exhibiting a single value of (Ω, μ) proves the solution is unstable, from the details of the equation for H^{tr} it can be seen there is a symmetry under the following transformation; $r_+ \to \alpha r_+$, $\Omega \to \Omega/\alpha$ and $\mu \to \mu/\alpha$ and so one can generate a range of values of (Ω, μ) for which the stability exists suggesting that it is both generic and robust.

4.3 Gubser-Mitra conjecture

Gubser and Mitra conjectured a precise relationship between the thermodynamics of black holes and their dynamical stability. They were partially motivated by the AdS/CFT correspondence; the low energy limit of a theory describing a single D-brane is ordinary electromagnetism, a $U(1)$ gauge theory. If we have N separated D-branes on any one of which an open string can end we have a $(U(1))^N$ gauge theory. If however the N D-branes are coincident the gauge group gets enhanced to $SU(N) \times U(1)$. Now D-branes are massive objects with a gravitational description, for small string coupling this is well described by classical supergravity. So, if the description of the branes on the one hand using gauge theory and on the other a theory of gravity are equivalent we are lead to conjecture a correspondence between the two. Motivated by the AdS/CFT correspondence black holes in asymptotically AdS space-times correspond to thermal states in the dual field theory. Gubser and Mitra's argument can then be summarised as follows; given a black brane solution we have a finite temperature dual field theory, for example the $2d$ conformal field theory Strominger and Vafa [49] used in the counting of microstates. If we identify the thermodynamical properties of the black hole solution with the dual field theory then a thermodynamic instability is in fact a thermodynamic instability in the field theory, an example of this would be the onset of a phase transition. As one nucleates a new phase in the field theory we find modes that grow exponentially in time, such a mode would then correspond to an exponentially growing mode in the gravity theory which signals a dynamical instability. Motivated by this Gubser-Mitra conjectured [62],

> "A black brane with a non-compact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable."

The ingredients of this conjecture can be split up as follows;

- *Non-compact translational symmetry*: This rules out the hyperspherical black hole solutions which are known to be classically stable and yet thermodynamically unstable [13]. For example the four-dimensional Schwarzschild solution of mass M has negative specific heat C

$$\begin{aligned} C &= \frac{\partial T_H}{\partial M} = -\frac{1}{8\pi M} && (4.12) \\ &< 0. && (4.13) \end{aligned}$$

  Also thermodynamic properties such as entropy contain information about long-wavelength physics which also suggests restricting to horizons that are infinite in size.

- *Thermodynamically unstable*: This is taken to mean that the Hessian of the entropy with respect to the other extensive thermodynamic quantities is negative definite[2].

- *Classical instability*: The existence of a tachyon mode in the linear fluctuation equations such as the one discovered by Gregory and Laflamme as reviewed in the previous section.

Gubser and Mitra provided evidence in favour of their conjecture by examining the stability of the Reissner-Nordström AdS_4 black hole solution, subject to numerical error their analysis supported their claim. Subsequently Reall [63] provided a partial proof in a different situation which is the subject of the next section.

[2]If there are no charged fields then this reduces to the requirement of negative specific heat, see previous bullet point. This fact is used on page 80

4.3.1 Reall's argument

Reall in [63] presented semi-classical evidence in favour of the conjecture in the case of translationally invariant black brane solutions. The basic points of his argument are given schematically in figure 4.1, each of which we shall briefly review.

- **1:** Gregory and Laflamme found that there exists a critical value μ_* of μ such that for every $\mu < \mu_*$ the perturbation equation has a solution regular at both the horizon and at spatial infinity. The values for the black string in various dimensions are tabulated in 4.2.3. The solutions with $\mu = \mu_*$ are independent of time and will be called *Lorentzian threshold unstable modes*, the step from the existence of an instability to the existence of such a threshold unstable mode is therefore mere definition. More generally we expect a time independent mode for the following reason; an unstable mode grows exponentially in time so the square of its energy must be negative, however a stable mode is oscillatory in nature and so the square of its energy is positive. Therefore the onset of instability occurs at zero energy so such a mode would be independent of time.

- **2:** The idea here is to form a relationship between Euclidean negative modes and modes for the Lichnerowicz operator with Lorentzian signature. First write the Euclidean eigenvalue equation as

$$\Delta_{LE} h_{\mu\nu}(x) = \lambda h_{\mu\nu}(x), \tag{4.14}$$

where Δ_{LE} is the Lichnerowicz operator for a metric with Euclidean signature and $\lambda < 0$ for a negative mode. Now for fluctuations in Lorentzian signature that are time independent we can Wick rotate to map between Δ_L and Δ_{LE}. The Lorentzian Lichnerowicz equation is

$$\Delta_L h_{\mu\nu}(x) = 0, \tag{4.15}$$

which at first sight doesn't look like equation (4.14), indeed this is how it should be since it is well known that the Schwarzschild solution has a Euclidean negative mode [66] and yet the Lorentzian solution is stable [13]. However if we

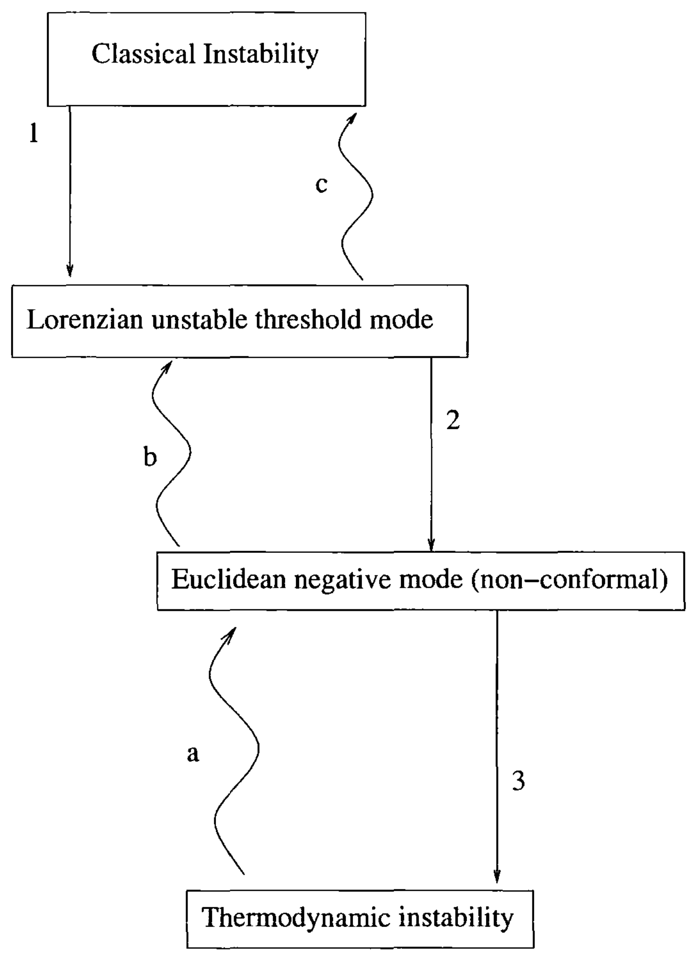

Figure 4.1: Schematic of Reall's argument.

now have a transverse extra dimension then we can decompose a fluctuation with momentum in the transverse space as

$$h_{\mu\nu}(x,z) = H_{\mu\nu}(x)e^{ikz}, \tag{4.16}$$

then if the extra dimension is translationally invariant[3] the Lichnerowicz equation can be written

$$\Delta_L H_{\mu\nu}(x) = -k^2 H_{\mu\nu}(x), \tag{4.17}$$

which is then, for a threshold unstable mode, mathematically equivalent to equation (4.14) with $\lambda = -k^2$.

For example the negative mode for the Euclidean Schwarzschild solution in four dimensions with mass $M = 1$ is $\lambda = -0.19$ [64], consequently we find that $k = 0.44$, which we are thinking of as corresponding to the threshold unstable value μ_*. This is in excellent agreement with the value in table 4.2.3. So we have motivated that a Lorentzian threshold unstable mode can be converted into a Euclidean negative mode by dropping the exp(..) factors and Wick rotating, although we still have to check that such a procedure preserves appropriate boundary conditions. The asymptotic behaviour is not affected by Wick rotation and so the Euclidean solution inherits this property from the Lorentzian one, regularity is also preserved on the horizon as explained in appendix A.3.

- **3:** This part of the argument requires a Euclidean negative mode to indicate the existence of a thermodynamic instability, this can be motivated by considering the Euclidean path integral approach for the canonical ensemble, specifically we consider

$$Z = \int d[g] \exp\left(-I[g]\right), \tag{4.18}$$

where I is the Euclidean Einstein-Hilbert action and the integral is taken over all Riemannian manifolds that are asymptotically flat. As usual, to make sense of this path-integral, we work with the semi-classical approximation,

[3] This is crucial here, the result doesn't hold without it. See chapter 5.

specifically we expand the action I around a classical solution and work to next to leading order. So writing

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{4.19}$$

we get

$$I[g] = I_0[\bar{g}] + I_2[\bar{g}, \delta g]. \tag{4.20}$$

The part of I_2 involving the traceless part of the metric perturbation, in transverse gauge, is proportional to

$$\int d^4x\sqrt{\bar{g}}H^{\mu\nu}\Delta_{LE}H_{\mu\nu}, \tag{4.21}$$

where $H_{\mu\nu}$ is transverse traceless and Δ_{LE} is the Euclidean Lichnerowicz operator. We evaluate the path integral over this part of the action by expanding $H_{\mu\nu}$ in terms of eigenfunctions of the Δ_{LE} operator, the contribution to the partition function then includes a factor of $\sqrt{\det \Delta_{LE}}$ so if Δ_{LE} has a negative eigenvalue there exists some pathology indicative of a thermodynamic instability of the canonical ensemble, there is however no known complete proof of the equivalence.

- **a:** To show the existence of a thermodynamic instability gives rise to a Euclidean negative mode we use the following line of reasoning. First construct a 1-parameter family of geometries for which the Euclidean action takes the following form

$$I = \beta E(x) - S(x), \tag{4.22}$$

where x is some parameter labelling the path. In [63] it is shown that this can be done for a large collection of solutions. We further require that for some value of x, say $x = T$ the geometry becomes the black solution of interest. At this point E and S are the energy and entropy of the solution respectively. Since the black hole extremises the action we must have that

$$\left(\frac{\partial I}{\partial x}\right)_{x=T} = 0. \tag{4.23}$$

If we further define

$$\frac{dE}{dS} = T(x), \tag{4.24}$$

then equation (4.23) implies that $T(x)$ is $1/\beta$ when $x = T$. Using this we find that

$$\left(\frac{\partial^2 I}{\partial x^2}\right)_{x=T} = \left[\frac{dT}{dx}\frac{d}{dT}\left(\frac{dE}{dx}\left(\beta - \frac{1}{T}\right)\right)\right]_{x=T} \tag{4.25}$$

$$= \left[\left(T\frac{dx}{dT}\right)^{-2}\frac{dE}{dT}\right]_{x=T}. \tag{4.26}$$

It now follows that if the solution is thermodynamically unstable, or in other words, if the specific heat dE/dT is negative the solution can't be a local minimum of the action. The action therefore can *decrease* as we move away from the solution along the path of geometries and therefore at least one eigenfunction must have a negative eigenvalue. This eigenvalue is exactly the Euclidean negative mode we are looking for.

- **b:** The recipe in this part is to simply take a Euclidean negative mode, Wick rotate and then multiply by $\exp(i\mu_i x^i)$ as is motivated in step 2 on page 63. It would appear that this would then give a Lorentzian threshold unstable mode as required. That this is indeed the case is not completely trivial and requires a detailed analysis of the perturbation equations to show that solutions to all the equations are indeed recovered in this way, for example the equations for the Euclidean negative mode don't "know" about the extra dimensions although these can be shown to be recovered correctly. The details of this last point are presented in [63].

- **c:** The existence of a threshold unstable mode is expected to separate stable short wavelength fluctuations of the black brane from unstable long wavelength ones indicating that a classical Gregory-Laflamme instability exists.

4.4 Charged p-branes

In this section we give a review of charged p-brane solutions in the context of this chapter. They provide a further non-trivial example of the relationship between thermodynamic and dynamic instability in favour of the Gubser-Mitra conjecture and are also necessary to discuss the more complicated smeared charged p-branes which we will use in the next chapter. It was mentioned in chapter 1 that there is evidence that the strong coupling limit of the various consistent string theories are described by an eleven-dimensional theory known as M-theory, of which the low energy dynamics are described by eleven-dimensional supergravity. The p-branes are solutions to a compactified version of this latter theory.

4.4.1 Explicit solutions

The compactified eleven-dimensional supergravity theory admits a consistent truncation to the following set of fields: the metric tensor g_{MN}, a scalar field ϕ and a field strength F_n of rank n. The action for these fields takes the following form

$$I = \int d^D x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!}e^{a\phi}F_n^2\right), \tag{4.27}$$

where a is a constant characteristic of the theory. Solutions for the theory described by this action are given by [74]

$$\begin{aligned} ds_D^2 &= \left(1 + \frac{k}{r^{\bar{d}}}\sinh^2\alpha\right)^{-\frac{4\bar{d}}{\Delta(D-2)}} \left(Udt^2 - \delta_{ij}dz^i dz^j\right) + \\ &\quad - \left(1 + \frac{k}{r^{\bar{d}}}\sinh^2\alpha\right)^{\frac{4d}{\Delta(D-2)}} \left(\frac{dr^2}{U} + r^2 d\Omega_n^2\right), \end{aligned} \tag{4.28}$$

$$e^{\eta\frac{\Delta}{2a}\phi} = 1 + \frac{k}{r^{\bar{d}}}\sinh^2\alpha, \qquad U = 1 - \frac{k}{r^{\bar{d}}}, \tag{4.29}$$

where;

- The field strength can carry either electric or magnetic charge, if ϵ_n is the volume form on a the unit n-sphere then the field strength is given by

$$F = \lambda * \epsilon_{D-n}, \qquad \text{or} \qquad F = \lambda\epsilon_n, \tag{4.30}$$

  respectively. In the former case we must have $\eta = 1$ and in the latter $\eta = -1$.

- $i = 1, \cdots, p$ so that the $\{z^i\}$ coordinates describe the p-dimensional spatial worldvolume, $\bar{d} = n - 1$, $d = p + 1$ and of course $D = 2 + n + p$.

- The parameter Δ is defined by $\Delta = a^2 + \frac{2d\bar{d}}{D-2}$. In general supersymmetric p-brane solutions can arise only when $\Delta = \frac{4}{N}$ where N is the number of field strengths participating in the solution, here of course $N = 1$.

- The constants α and k are related to the charge and mass of the solution respectively, the precise relationship is given in the next section.

4.4.2 Thermodynamics/Stability

The mass M, charge Q, temperature T and entropy S of this solution are given by [71]

$$M = k\left(\bar{d}+1+\frac{4\bar{d}}{\Delta}\sinh^2\alpha\right), \quad \text{and} \quad Q = \frac{\bar{d}k}{\sqrt{\Delta}}\sinh 2\alpha \tag{4.31}$$

$$T = \frac{\bar{d}}{4\pi r_+}(\cosh\alpha)^{-\frac{4}{\Delta}}, \quad S = \frac{1}{4}r_+^n\Omega_n(\cosh\alpha)^{\frac{4}{\Delta}} \tag{4.32}$$

where $r = r_+$ is the location of the horizon and Ω_n is the volume of a unit n-sphere. The condition that the entropy is a local maximum in its thermodynamic phase space is equivalent to the condition that

$$C_Q = \left(\frac{\partial M}{\partial T}\right)_Q > 0 \quad \text{and} \quad \left(\frac{\partial \Phi_H}{\partial Q}\right)_T > 0, \tag{4.33}$$

where Φ_H denotes the charge potential energy at the horizon. Explicitly we find that

$$C_Q = -4\pi r_H^{\bar{d}+1}(\cosh\alpha)^{\frac{4}{\Delta}}\frac{2\bar{d}+(\Delta+\bar{d}(\Delta-2))\cosh 2\alpha}{2\bar{d}+(\Delta-2\bar{d})\cosh 2\alpha}, \tag{4.34}$$

$$\left(\frac{\partial \Phi_H}{\partial Q}\right)_T = \frac{r_H^{-\bar{d}}\Delta\cosh 2\alpha}{2\bar{d}+(\Delta-2\bar{d})\cosh 2\alpha}. \tag{4.35}$$

Now since the term $\Delta+\bar{d}(\Delta-2)$ is positive definite the condition for thermodynamic stability is impossible to achieve, however[4] as there is no charged field to carry the charge we will take the condition of thermodynamic stability to be simply the positivity of C_Q. In which case we find that the specific heat is always negative if

$$|a| \geq \frac{D-3-p}{\sqrt{(D-2)/2}} = a_{cr}. \tag{4.36}$$

as this guarantees that $\Delta - 2\bar{d} \geq 0$. If we further define α_{cr} to satisfy the equation

$$\sinh^2\alpha_{cr} = \frac{2(D-3-p)(p+1)+(D-2)a^2}{2\left(2(D-3-p)^2-(D-2)a^2\right)}, \tag{4.37}$$

[4]Following [63, 65, 71].

then for $|a| < a_{cr}$ the specific heat is negative if $0 \leq \alpha < \alpha_{cr}$ and positive otherwise. At the critical value α_{cr} the specific heat diverges.

For example;

- The Dp-branes of type II string theory have (for D=10) $a = 1/2(p-3)$ and so for $p \geq 5$ the solutions are always thermodynamically unstable. For $p < 5$ the specific heat changes sign for some critical value of the charge.
- The black p-branes considered by Gregory and Laflamme have $a = 1/2(1-n)$ which for $D = 10$ implies that $a = a_{cr}$ implying that the specific heat is always negative.

The second point above is in complete agreement with the Gubser-Mitra conjecture since Gregory and Laflamme found those solutions to be classically unstable. In [71] a perturbation analysis was performed for the black p-branes solutions given in this section with added generality of allowing for a variety of values of a. The authors looked for solutions of the type discussed in section 4.2.1, namely those which are spatially regular but which grow exponentially in time, their findings are in remarkable agreement with the Gubser-Mitra conjecture. Specifically they found that the threshold mass for a regular perturbation (see section 4.2.3) goes to zero when $\alpha = \alpha_{cr}$, i.e. when the specific heat changes sign. In other words the Gregory Laflamme type instability that was present when the solution was thermodynamically unstable doesn't persist into the region of thermodynamic stability.

The data in table 4.2, taken from [71], gives an example of the closeness of the relationship in a special case.

p	1	2	3	4	5	6
Numerical	0.418	0.549	0.695	0.881	1.178	>4
Gubser-Mitra	0.4186	0.5493	0.6954	0.8814	1.1791	∞

Table 4.2: The first row of data in the table shows the numerical value of α for which the threshold mass goes to zero for various p-branes in 10-dimensions with $a = 1/2$, the second row shows the value of α_{cr}.

Chapter 5

On The Gubser-Mitra conjecture

5.1 Introduction

As mentioned in the introduction of this thesis the existence of an instability raises two interesting questions, one being can we identify a criterion to establish easily if any given black brane solution is unstable and the other being what is the end state of such an instability if it indeed exists in the first place. The Gubser-Mitra conjecture discussed in the previous chapter addresses the first question and so far all the evidence is in support of it. In this chapter we will show how work on *non-uniform* solutions of the supergravity action

$$I = \int d^D\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2(p+2)!}e^{a\phi}F_{p+2}^2\right), \tag{5.1}$$

leads us to a contradiction of this conjecture [67].

Let's first go back to the original paper of Gregory and Laflamme [60] discussed in chapter 4, they suggested that a black string solution wrapping a cylinder would break up and form a black hole, as was also mentioned in chapter 1 this outcome is at least entropically favourable. However Horowitz and Meada [78] subsequently have shown that this is not possible, any $\mathbb{S}^1$ around the horizon can't shrink to zero size in finite affine parameter, so the classical evolution of an instability can't change the horizon topology, in other words a black string can't pinch off. This suggests that there are new solutions which are not uniform in the compact direction, a conclusion which was supported by numerical evidence provided by Wiseman [80].

Harmark and Obers derived a simple ansatz for such non-uniform solutions and although they did not find any explicit solutions (as we have already mentioned in section 4.1 the geometry is expected to be very complicated) the ansatz is very useful. We will use their ansatz to argue that the Gubser-Mitra conjecture fails for charged smeared branes, i.e. charged branes which have an extended direction *not* coupled to the charge. It would be natural to ask at this point where Reall's argument presented in chapter 4 breaks down. The most obvious problem is with the simple relationship between the Lichnerowicz operator on the space-time, Δ_L, and the Euclidean operator, Δ_{LE}, explained in section 4.3.1. If there are non-trivial components of the metric associated with directions transverse to the brane then the simple eigenvalue equation (c.f. equation (4.17))

$$\Delta_L H_{\mu\nu}(x) = -k^2 H_{\mu\nu}(x), \tag{5.2}$$

picks up extra position dependence on the RHS and so such solutions can't be identified with negative Euclidean eigenmodes in any simple way.

The rest of this chapter is organised as follows; first we give a discussion on the phase diagrams we will need in section 5.5, then a review of Harmark and Obers' ansatz for non-uniform p-brane solutions and then finally how to use everything to argue the existence of a counter-example to the Gubser-Mitra conjecture.

5.2 Phase structure

Before we give a discussion of Harmark and Obers' ansatz and how we can use it to study the Gubser-Mitra conjecture we first need to explain how the known neutral and static black string solutions can be gathered together into a single phase diagram introduced in [87], this phase diagram will be used in the arguments in section 5.5.

To be specific we will consider static and neutral black holes on the cylinder $\mathbb{R}^{d-1} \times \mathbb{S}^1$. For static and neutral mass distributions in flat space $\mathbb{R}^d$ the asymptotic leading order correction to the metric is given by the mass, on a cylinder however we need an additional quantity called the binding energy; intuitively if we have a band wrapped around a circle with some tension then the tension will receive self gravity contributions if it is massive enough, these contribute to the total tension of the system. The flat metric on the cylinder will be taken to be

$$ds_{d+1}^2 = dt^2 - dz^2 - dr^2 - r^2 d\Omega_{d-2}^2, \tag{5.3}$$

where z is a periodic coordinate with period $2\pi R_T$. We will consider metrics which asymptotically take this form and, in addition, with leading order asymptotic behaviour independent of z - this last requirement amounts to the physical assumption that far from any mass distributions the physical dependence on the periodic coordinate should vanish. It is shown in [87] that with these assumptions asymptotically two of the metric components necessarily have leading order behaviour

$$g_{00} = 1 - \frac{c_t}{r^{d-3}}, \qquad g_{zz} = -1 - \frac{c_z}{r^{d-3}}, \tag{5.4}$$

and that c_z and c_t contain gauge invariant information about the metric. In particular the mass M and relative binding energy n are given in terms of them by

$$M = \frac{\Omega_{d-2} 2\pi R_T}{16\pi G_N}\left((d-2)c_t - c_z\right), \qquad n = \frac{c_t - (d-2)c_z}{(d-2)c_t - c_z}. \tag{5.5}$$

Note that in this last equation we have explicitly written $\kappa = 8\pi G_N$ in the notation of equation (2.23) of chapter 2. These two independent quantities can be used as variables in a two-dimensional phase diagram for neutral and static black hole solutions and we can partially construct this phase diagram using solutions that are already known either explicitly or as a result of numerical investigations.

First we will consider the black hole solutions, we know that such a branch in the phase space would start at $(M, n) = (0, 0)$ and although the exact behaviour of these solutions is unknown away from the origin we know that the branch would have to terminate at some large value of the mass M as the horizon would meet itself around the cylinder. A second branch is given by the uniform neutral black string solutions, which exist for arbitrary mass. These are the solutions obtained by taking the direct product of a Schwarzschild black hole solution with a circle. In $d+1$ dimensions the metric can be written as,

$$ds_{d+1}^2 = f dt^2 - f^{-1} dr^2 - dz^2 - r^2 d\Omega_{d-2}^2 \tag{5.6}$$

where $f = 1 - (r_0/r)^{d-3}$. So from the definition of the binding energy we infer that

$$n = \frac{1}{d-2}. \tag{5.7}$$

In the phase diagram for fixed dimensionality this family of solutions appears simply as a horizontal line. Finally there is a family of solutions discovered numerically by Wiseman [80]. Gubser [79], assuming directions transverse to the brane remain rotationally symmetric, introduced a non-uniformity parameter λ defined by

$$\lambda = \frac{1}{2}\left(\frac{R_{max}}{R_{min}} - 1\right), \tag{5.8}$$

where R_{max} and R_{min} are the maximum and minimum values of the Schwarzschild radius of the solution. For example for the uniform solution in equation (5.6) we have $\lambda = 0$. Wiseman [80], by using relaxation techniques with a translationally invariant black string solution of critical mass as an initial guess, numerically found non-uniform solutions for values of λ up to nine. His numerical data was translated into the M and n variables of our phase diagram in [87] and these data are also plotted.

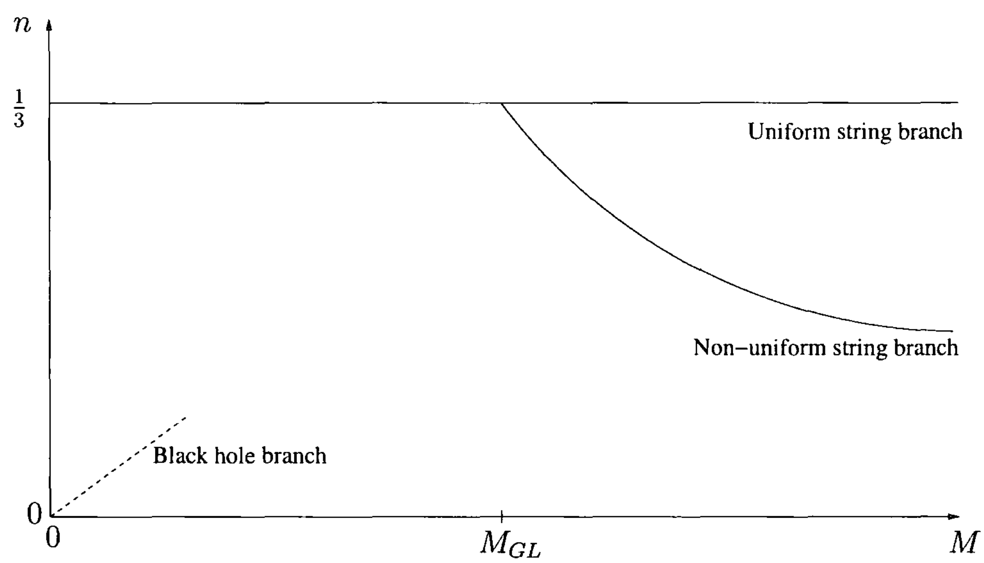

Figure 5.1: Phase diagram for neutral solutions for a five-dimensional system on a circle.

5.3 The ansatz

In [83] Harmark and Obers presented an ansatz for electrically charged dilatonic black p-brane solutions on a cylinder $\mathbb{R}^{d-1} \times S^1$. The ansatz was motivated by introducing a new coordinate system which interpolates between the usual black brane with transverse space $\mathbb{R}^d$ (c.f. section 4.4.1)

$$\begin{aligned} ds_D^2 = &\ H^{-\frac{d-2}{D-2}}\left(fdt^2 - \textstyle\sum_{i=1}^p (dx^i)^2 - H\left(f^{-1}d\rho^2 + \rho^2 d\Omega_{d-1}^2\right)\right) \\ & e^{2\phi} = H^a, \qquad A_{01\ldots p} = \coth\alpha(1 - H^{-1}), \end{aligned} \tag{5.9}$$

where

$$f = 1 - \frac{\rho_0^{d-2}}{\rho^{d-2}}, \qquad H = 1 + \frac{\rho_0^{d-2}\sinh^2\alpha}{\rho^{d-2}}, \tag{5.10}$$

which is a good description of a black hole on a cylinder of small mass, as in such a situation the black hole has a small radius and in some sense doesn't know that the transverse space is compact, and the same black brane smeared over the transverse circle

$$\begin{aligned} ds_D^2 = &\ H^{-\frac{d-2}{D-2}}\left(fdt^2 - \textstyle\sum_{i=1}^p (dx^i)^2 - H\left(f^{-1}dr^2 + dz^2 + r^2 d\Omega_{d-2}^2\right)\right) \\ & e^{2\phi} = H^a, \qquad A_{01\ldots p} = \coth\alpha(1 - H^{-1}), \end{aligned} \tag{5.11}$$

where

$$f = 1 - \frac{r_0^{d-3}}{r^{d-3}}, \qquad H = 1 + \frac{r_0^{d-3}\sinh^2\alpha}{r^{d-3}}, \tag{5.12}$$

which is a good description at large mass since in this case the horizon radius is large enough to fully explore the compact direction.

The ansatz in the new coordinates, given in [83], is

$$ds_D^2 = H^{-\frac{d-2}{D-2}}\left(fdt^2 - \sum_{i=1}^p (dx^i)^2 - HR_T^2\left(f^{-1}AdR^2 + \frac{A}{K^{d-2}}dv^2 + KR^2 d\Omega_{d-2}^2\right)\right) \tag{5.13}$$

$$e^{a\phi} = H^2, \quad A_{01..p} = \coth\alpha\left(1 - H^{-1}\right), \tag{5.14}$$

$$f = 1 - \frac{R_0^{d-3}}{R^{d-3}}, \quad H = 1 + \frac{R_0^{d-3}\sinh^2\alpha}{R^{d-3}}, \tag{5.15}$$

where v is periodic with period 2π and A and K are two unknown functions of R and v only. The total space-time dimension $D = d + p + 1$ and the ansatz is for a solution with an event horizon located at $R = R_0$.

5.4 Analysis

Given the ansatz in the previous section an important question is; are the resulting equations of motion consistent? The next subsection deals with this issue. Section 5.4.2 then points out an important property of the the equations of motion for the unknown fields which will be exploited in section 5.5.

5.4.1 Consistency

The ansatz given in the previous section is an ansatz for solutions to the equations of motion derived from the action (5.1), namely

$$R_{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}e^{a\phi}F_{\lambda\mu}F^{\lambda}_{\ \nu} = g_{\mu\nu}\left(\tfrac{1}{4(2-n)}\right)e^{a\phi}F^2, \tag{5.16}$$

$$\nabla^2\phi = \tfrac{a}{4}e^{a\phi}F^2, \qquad \nabla_\mu\left(e^{a\phi}F^{\mu\nu}\right) = 0. \tag{5.17}$$

One of the equations of motion can be solved algebraically for A in terms of K; this leaves a system of three second-order equations which need to be satisfied by the function $K(R, v)$. Generically, such a system is heavily over-determined; however, it was shown in [83] that the system is consistent to second order in perturbation theory. This surprising result was elucidated in [84, 85], where it was shown that in the neutral case, the seemingly restricted ansatz taken above is in fact equivalent to the most general ansatz consistent with the symmetries. For the neutral black string, the Harmark and Obers ansatz reduces to

$$ds^2_{d+1} = fdt^2 - R_T^2\left(f^{-1}AdR^2 + \frac{A}{K^{d-2}}dv^2 + KR^2d\Omega^2_{d-2}\right), \tag{5.18}$$

where

$$f = 1 - \frac{R_0^{d-3}}{R^{d-3}}. \tag{5.19}$$

Using staticity and spherical symmetry, the most general metric for a black string can be brought to the form

$$ds^2_{d+1} = e^{2B}dt^2 - e^{2C}\left(dr^2 + dz^2\right) - e^{2D}d\Omega^2_{d-2}, \tag{5.20}$$

where z is a periodic coordinate of period $2\pi R_T$ and $r \geq 0$ with $r = 0$ being the location of the horizon. This is referred to as the conformal form, as the (r, z) space is written in conformally flat coordinates. Here B, C and D are functions of r and z only. Since the metric in (5.20) involves three arbitrary functions, while that in (5.18) only involves two, it seems like the former must be more restrictive. Of course from a purely geometrical point of view the former *is* more restrictive, however with the additional constraint of having the equations of motion satisfied they are in fact equivalent [84,85]. We can see this as follows, first perform the following coordinate transformations and redefinitions in equations (5.18)

$$\begin{aligned} R^{d-3} &= R_0^{d-3} + \bar{r}^{d-3}, & (5.21)\\ \hat{A} &= f^{-1} A R_T^2 \left(\frac{\bar{r}}{R}\right)^{2(d-4)}, & (5.22)\\ \hat{K}^{d-2} &= \frac{K^{d-2}}{f}\left(\frac{\bar{r}}{R}\right)^{2(d-4)}. & (5.23)\end{aligned}$$

We find that the metric takes the following form

$$ds_{d+1}^2 = \frac{\bar{r}^{d-3}}{R_0^{d-3} + \bar{r}^{d-3}} dt^2 - \hat{A}\left(d\bar{r}^2 + \frac{dv^2}{\hat{K}^{d-2}}\right) - R_T^2 \hat{K} \bar{r}^{\frac{5-d}{d-2}} \left(R_0^{d-3} + \bar{r}^{d-3}\right)^{\frac{3}{d-2}} d\Omega_{d-2}^2. \tag{5.24}$$

Next write $\hat{A} = e^{2a}$ and $R_T^2 \hat{K} = e^{2k}$ then transform to the conformal form by making the following transformation

$$\bar{r} = g(r,z), \qquad v = h(r,z) \tag{5.25}$$

$$\partial_r g = e^{-(d-2)k} \partial_z h, \qquad \partial_z g = e^{-(d-2)k} \partial_r h, \tag{5.26}$$

where the last two equations are a sufficient condition for $drdz$ cross-terms to vanish. We can bring the ansatz for the metric in the neutral case (5.18) into the conformal form (5.20) if

$$g^{d-3} = \frac{R_0^{d-3} e^{2B}}{1 - e^{2B}}, \tag{5.27}$$

and

$$e^{2a} = \frac{e^{2c}}{(\partial_r g)^2 + (\partial_z g)^2}, \qquad e^{2k} = \frac{R_T^2}{R_0^2} e^{2D} e^{\frac{2(d-5)}{(d-2)(d-3)} B} \left(1 - e^{2B}\right)^{\frac{2}{d-3}}. \tag{5.28}$$

The system of equations in (5.26) imply an integrability condition which together with (5.28) implies

$$(\partial_r^2 + \partial_z^2) B + (\partial_r B)^2 + (\partial_z B)^2 + (d-2)(\partial_r B \partial_r D + \partial_z B \partial_z D) = 0, \tag{5.29}$$

but this is exactly the $R_{tt} = 0$ equation of motion for the conformal metric (5.20). This means that the ansatz in the neutral case is *consistent*. To show that the ansatz is in fact the most general possibility with the assumptions we have imposed, we also have to check that the location of the horizon and the periodicity of the compact direction are reproduced correctly. In the conformal form these are $\bar{r} = 0$ and $z \to z + 2\pi R_T$ respectively. The location of the horizon is straightforward to obtain, when $r = 0$ we have that e^B vanishes and so by equation (5.27) $g(0, v)$ must vanish also, but this in turn means that $\bar{r} = 0$ and hence from equation (5.21) $R = R_0$, so $r = 0$ corresponds exactly to $R = R_0$ as required. The periodicity on the other hand is slightly harder, v is taken to have periodicity 2π and so we want that $z \to z + 2\pi R_T$ as $v \to v + 2\pi$. Looking at equation (5.25) this is clearly the case if

$$h(r, z + 2\pi R_T) - h(r, z) = 2\pi, \tag{5.30}$$

now the RHS of equation (5.27) is periodic in z and therefore so is $g(r, z)$, this in turn means that $\partial_z g(r, z)$ is also periodic in z and so from equation (5.26) $\partial_r g(r, z)$ is as well. The point here then is that this proves that the LHS of equation (5.30) is independent of r as required. We still have to show that the correct value of 2π is obtained on the RHS, to this end we note that the LHS of equation (5.30) can be written, using equation (5.26), as

$$\int_0^{2\pi R_T} dz \partial_z h = \int_0^{2\pi R_T} dz e^{(d-2)k} \partial_r g, \tag{5.31}$$

however since this equation is independent of r we can consider it in the limit as $r \to \infty$, this is a useful thing to do for two reasons, first because in such a case $k \to 0$ and secondly because asymptotically the metric takes the following form

$$g_{00} = 1 - \frac{c_t}{r^{d-3}}, \tag{5.32}$$

and so $1 - e^{2B} = c_t r^{-(d-3)}$ to leading order. If we use this in equation (5.27) we obtain that equation (5.30) is true provided that $c_t = (R_T R_0)^{d-3}$ which is indeed the case in the R coordinate system since $r/R \to R_T$ asymptotically.

This proves that the ansatz (5.18) is in fact the most general possibility. The extent to which this argument may be generalised to the charged case is given in appendix A.4.

5.4.2 Charge dependence

An important point regarding the resulting system of equations following from equations (5.16) and (5.17) for A and K for the general ansatz is that they are independent of the charge (i.e. of α), and hence also of the value of p (since the extra dimensions x^i decouple in the neutral case). Furthermore, the boundary condition necessary to ensure regularity at the horizon is simply that $A(R_0, v)$ and $K(R_0, v)$ are constants, so the boundary conditions also do not involve the charge, this boundary condition corresponds physically to requiring that the surface gravity, and hence the temperature (see chapter 4), is constant along the horizon. Importantly this allows us to map the problem of finding a charged solution of the form (5.13) to finding a solution in the uncharged case.

5.5 Counter-example

We can now provide a counter-example to the Gubser-Mitra conjecture by using the ansatz (5.15). The uniform smeared black p-brane is given by setting $A = K = 1$, its thermodynamics are equivalent to those of the T-dual $p+1$-brane solution [88] and so can be computed from the results in our discussion of p-brane solutions in section 4.4 of chapter 4. In particular, the mass and charge are[1]

$$M = \frac{\Omega_{d-2} 2\pi R_T V_p}{16\pi G} (R_T R_0)^{d-3} [(d-2) + (d-3)\sinh^2\alpha], \tag{5.33}$$

$$Q = \frac{\Omega_{d-2} 2\pi R_T}{16\pi G} (R_T R_0)^{d-3} (d-3) \sinh\alpha \cosh\alpha, \tag{5.34}$$

while the entropy and temperature are

$$S = \frac{\Omega_{d-2} 2\pi R_T V_p}{4G} (R_T R_0)^{d-2} \cosh\alpha, \quad T = \frac{d-3}{4\pi (R_T R_0)\cosh\alpha}. \tag{5.35}$$

The statement of the Gubser-Mitra conjecture uses the Hessian matrix of derivatives of the entropy as the test for thermodynamic stability. However, following [63,71,72], we will assume that there is no charged field in the theory, so the charge is not

[1]We take the v and x^i coordinates to be periodically identified to allow us to write finite expressions.

allowed to vary as a function of position. This assumption in particular allows us to focus just on the specific heat when considering the thermodynamics (if the charge is not able to redistribute itself there is no way it can be responsible for altering the entropy): we take the condition for thermodynamic stability then to simply be the positivity of the specific heat

$$C_Q = \left(\frac{\partial M}{\partial T}\right)_Q > 0. \tag{5.36}$$

We can then calculate from the formulae (5.34) and (5.33) that the specific heat is negative at $Q = 0$ for all values of d, in fact in this case we find that

$$C_0 = -(d-2)(d-3)\frac{\Omega_{d-2}2\pi R_T V_p}{16\pi G}\left(\frac{d-3}{4\pi}\right)^{d-3} T^{2-d}, \tag{5.37}$$

but it becomes positive above some critical charge if $d > 5$ as near extremality we find that

$$M \approx Q + \left(\frac{\Omega_{d-2}2\pi R_T V_p}{16\pi G}\right)^{-\frac{2}{d-5}} \left(\frac{16\pi^2}{(d-3)^3}\right)^{\frac{d-3}{d-5}} (d-2)(T^2 Q)^{\frac{d-3}{d-5}}. \tag{5.38}$$

We now reiterate the key feature of non-uniform solutions in this ansatz. In [83], it was shown that when we impose the equations of motion[2]

$$R_{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}e^{a\phi}F_{\lambda\mu}F^{\lambda}{}_{\nu} = g_{\mu\nu}\left(\tfrac{1}{4(2-n)}\right)e^{a\phi}F^2, \tag{5.39}$$

$$\nabla^2\phi = \tfrac{a}{4}e^{a\phi}F^2, \qquad \nabla_\mu\left(e^{a\phi}F^{\mu\nu}\right) = 0, \tag{5.40}$$

the solution is *independent* of the charge. The important point for our present purpose then is that this implies that any solution of the equations of motion describing a neutral black string, uniform or non-uniform, can be written in the form (5.18). This provides a convenient framework for discussing solutions. To include these solutions on the phase diagram we have already discussed we need the mass M and the relative binding energy n. For general charge, the mass is

$$M = \omega(R_T R_0)^{d-3}[(d-2) + (d-3)\sinh^2\alpha] \tag{5.41}$$

and the binding energy parameter is

$$n = \frac{1-(d-2)(d-3)\chi}{(d-3)\sinh^2\alpha + (d-2) - (d-3)\chi}, \tag{5.42}$$

[2] Derived from the action in (5.1).

where

$$\omega = \frac{\Omega_{d-2} 2\pi R_T}{16\pi G_N}, \tag{5.43}$$

and χ parametrises the asymptotic falloff of the unknown function K [83], and is hence independent of the charge.

The known phase structure for five-dimensional neutral static solutions on a circle of radius R_T is sketched in figure 5.1. Now since the equations of motion are independent of the charge, each solution in figure 5.1 gives a solution for every value of the charge. Inspection of (5.41,5.42) shows that adding charge increases the mass, as expected, and decreases n, enhancing the binding effect. Thus, if we plot M vs n in the charged case, we get a qualitatively similar picture, as shown in figure 5.2. We can see that on a circle of some fixed radius R_T, there is always some critical

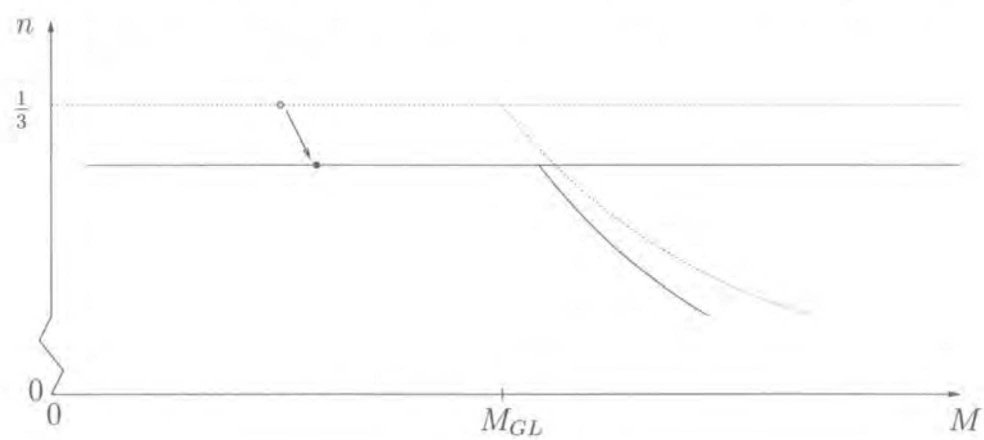

Figure 5.2: The bold lines now refer to charged solutions. The diagram is qualitatively the same as in the neutral situation, shown as dotted lines for reference.

value of the mass at which a non-uniform branch joins on to the uniform smeared black hole branch. We can re-state this in terms relevant for the Gubser-Mitra conjecture: for any given value of the mass and charge, there is a finite wavelength at which a threshold unstable mode occurs. We know that for zero charge, the uniform black string is unstable to modes of longer wavelength. Although we have not demonstrated the existence of the corresponding dynamical instability explicitly in the charged case, the persistence of the threshold unstable mode is strong evidence that it exists.

This result should be contrasted with the analysis of p-brane solutions in [71–73] discussed in chapter 5. In those studies, it was found that for the ten-dimensional supergravity p-brane solutions with $p \leq 4$, there is a threshold unstable mode for the neutral case, but this mode goes off to infinite wavelength at a critical value of the charge, signalling the disappearance of the instability. This was found to occur at the same critical value of the charge where the specific heat changes sign.

In our case, by contrast, the threshold unstable mode exists all the way up to extremality, even though the specific heat changes sign before we reach extremality for cases with $d > 5$. Thus, there are smeared branes which are locally thermodynamically stable, but possess a dynamical instability by the arguments of this section. This is a clear violation of the Gubser-Mitra conjecture. It is interesting to note that even though we have pointed out a possible failure of Reall's argument for this situation it is clear from the result in this section that the

$$\text{Classical instability} \rightarrow \text{Thermodynamic instability} \tag{5.44}$$

part of the argument *must* fail. Note also that the wavelength of the threshold unstable mode, which signals the onset of instability, is determined by R_0, since the equations for K are independent of charge. Hence if we go near extremality by taking $R_0 \rightarrow 0$ and $\alpha \rightarrow \infty$ keeping M fixed, the wavelength of the unstable mode will go to zero, suggesting that the instability will appear sufficiently close to extremality for any compactified black string as well.

Chapter 6

Conclusions

This thesis has discussed two modern aspects of theoretical physics, braneworlds and black holes. From a modern point of view the physics of each are closely related, here we briefly review each in turn.

6.1 Braneworlds

Physical theories in higher-dimensional space-times can have rich structures, Kaluza-Klein theory arose as a result of trying to use them to unify electromagnetism with general relativity. Kaluza-Klein theory now provides a useful tool in modern physics as it allows us to sweep away, in a systematic fashion, higher dimensions of space-time which seem necessary when considering unified theories such as superstring theory and M-theory which require ten and eleven dimensions respectively. Kaluza-Klein theory hides the extra dimensions by supposing that they are compact. Braneworlds on the other hand deal with higher dimensions in a fundamentally different way, the Randall-Sundrum models discussed in chapter 2 provide us with an "alternative to compactification" - in other words they do not rely on the Kaluza-Klein doctrine. This offers us both a novel alternative and a richer theoretical base of study, moreover they are motivated by the higher-dimensional theories themselves.

The Randall-Sundrum models are primarily models in five-dimensional space-times, they consist of branes which have codimension 1. Of course if we have consistent theories in even higher-dimensional spaces then we could use a combination of techniques adapted from both Kaluza-Klein and Randall-Sundrum models to obtain more realistic scenarios. However there is another possibility; consider braneworlds with *higher* codimension. Higher codimension models - specifically codimension 2 - had already been studied in the literature, it was found that in Einstein gravity such models were over-restrictive on the matter and energy that such a brane could support and is a clear example of how bulk geometry restricts worldvolume fields. As explained in chapter 3 such a "no-go" result can be circumvented if we generalise Einstein's equations to the Einstein-Gauss-Bonnet equations which are motivated by both string theory and pure generality required by the geometry of higher-dimensional space-times. Specifically we showed that the theory on a codimension 2 braneworld could be at least as diverse as the usual Einstein theory. The main result of chapter 3 is summarised in the following equation of motion for the braneworld metric $\hat{g}_{\mu\nu}$

$$\boxed{2\pi(1-\beta)M^4\left[\hat{g}_{\mu\nu} + 4\alpha\hat{G}_{\mu\nu} + \alpha\frac{\beta}{1-\beta}\hat{W}_{\mu\nu}\right] = \hat{T}_{\mu\nu}} \tag{6.1}$$

where the braneworld energy-momentum $\hat{T}_{\mu\nu}$ and the Weyl like tensor $\hat{W}_{\mu\nu}$ are defined in chapter 3. Examples and consequences of this equation are also discussed in chapter 3, we find that in the simplest cases the usual Einstein theory is reproduced yet, with the addition of the Weyl like terms $\hat{W}_{\mu\nu}$, there could be interesting modifications - at the least we have some of the success of the Randall-Sundrum models. It is however not clear that in our model an observer would measure the usual Newtonian gravitational law to first order as is possible to motivate in the Randall-Sundrum models. We were able to show that the braneworld components of the perturbation equations can be written as

$$\boxed{4\alpha\Delta\overline{\delta R}_{\mu\nu} = \overline{\delta T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{\delta T} + \frac{1}{2}g_{\mu\nu}\Delta\phi} \tag{6.2}$$

where $\overline{\delta R}_{\mu\nu}$ is the usual Lichnerowicz operator calculated using the four-dimensional braneworld metric. This result is promising, it avoids the problem in six dimensions where we find that Einstein's equations yield

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T, \tag{6.3}$$

the factor of $1/4$ being evidence of two extra dimensions.

Since the main result in chapter 3 there has been a lot of interest in these models to attack the cosmological constant problem using the automatic relationship between the brane tension and the deficit angle [40, 42], to construct models by intersecting branes [41] and also to study the effect of Gauss-Bonnet terms in the more familiar codimension one scenarios [55]. There has also been further investigation into the origin of the no-go result of Cline et. al. [56] using *thick* branes instead of resorting to Gauss-Bonnet terms as we used in [53]. In fact in [42] they find that to lowest order in the density of matter on the brane the usual four-dimensional FRW cosmology is reproduced, there would however be higher order corrections which have not yet been calculated. There is clearly still much interesting work to do here.

6.2 Black Holes

Once it is appreciated that certain black hole solutions in higher dimensions are unstable there are a number of interesting directions to pursue as mentioned in the introduction in chapter 1. Knowing when a solution is stable is obviously an important physical question to which the second half of this thesis is devoted. The identification of the laws of black hole mechanics with the usual laws of thermodynamics motivated a partial solution. The Gubser-Mitra conjecture, motivated and discussed in chapter 4, reads

> "A black hole with a non-compact translational symmetry is classically stable if and only if it is thermodynamically stable."

As also discussed in chapter 4, a partial proof of this conjecture for certain translationally symmetric black p-brane solutions can be given along with further evidence in the results of non-trivial numerical investigations for other more general solutions. However a related development of Horowitz and Maeda that black brane solutions would not be able to change their horizon topology through some classical evolution led Harmark and Obers to focus on solutions which would not have the translational invariance used by Reall in his argument, to this end Harmark and Obers presented a consistent ansatz for non-uniform solutions possibly smeared over a compact direction which crucially was independent of a charge parameter. This ansatz is used in chapter 5, where we explain how it motivates the existence of solutions which possess instabilities of the Gregory-Laflamme type but which are also thermodynamically stable and hence provide a counter-example to the Gubser-Mitra conjecture as it is stated above.

A natural question to ask is of course where their heuristic argument for the instability fails or in other words to understand how the smeared charge affects the thermodynamics from the dual field theory point of view. Such an understanding would surely indicate how to modify the conjecture in light of these results and give a deeper understanding of the connection between the two distinct types of instability and, more subtly, how they are related. The end state of the instability is also unresolved, the non-uniform solutions found by Wiseman [80] can't be the end state of the Gregory-Laflamme instability as they have too large a mass. Higher-dimensional black holes have such a rich structure that it would be interesting to more fully understand this.

It is also worth mentioning the possibility of studying black hole thermodynamics in Einstein-Gauss-Bonnet gravity, a recent discussion of this is given in [109]. In this paper the authors offer a Noether charge approach to computing the entropy of black hole solutions with Gauss-Bonnet corrections, in particular they point out that the resulting corrections are interestingly reminiscent of our results in chapter 3. To be specific the entropy S for a solution in n dimensions is found to be

$$S \sim \int d^{n-2}x\sqrt{h}\,(1+2\alpha R)\,, \tag{6.4}$$

where $h_{\mu\nu}$ and R are the metric and scalar curvature on the horizon respectively. It would be very interesting to understand the relation between the two results.

The current state of physics provides much hope of understanding quantum gravity, non-trivial and unforeseen connections between various aspects of current theories lead us on. There is however still lots of be understood.

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Appendix A

Miscellaneous details

A.1 Gauss-Codazzi formalism

The Gauss-Codazzi formalism provides a powerful tool for describing the geometry of the embedding of a submanifold $\mathcal{S}$ in a manifold $\mathcal{M}$.

Let us work in complete generality. Suppose that $\mathcal{M}$ is a n-dimensional manifold with coordinates x^a $(a = 0, ..., n-1)$[1] and that $\mathcal{S}$ is a d-dimensional submanifold. We want to describe how $\mathcal{S}$ bends and contorts in $\mathcal{M}$.

To do this with a simple one-dimensional curve in a three-dimensional Euclidean space parametrised by arc length s we use the well known Serret-Frenet formulae [2]

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \tag{A.1.1}$$

$$\frac{d\mathbf{n}}{ds} = -\tau\mathbf{b} - \kappa\mathbf{t}, \tag{A.1.2}$$

$$\frac{d\mathbf{b}}{ds} = \tau\mathbf{n}, \tag{A.1.3}$$

where $\mathbf{n}, \mathbf{t}$ and $\mathbf{b}$ are the normal, tangent and binormal vectors respectively and κ and τ are the curvature and torsion. The curvature of the curve can be thought of as the rate we "pull away" from the normal at each point as we move along it, in the case of a plane curve this is easy to see.

[1]This is a more convenient convention here, there is no confusion with results in the main text.

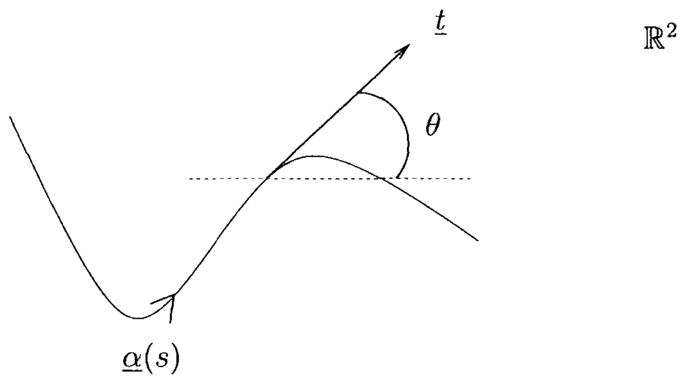

Figure A.1: .

In the diagram we have $\mathbf{t} = (\cos\theta, \sin\theta)$ so $\mathbf{n} = (-\sin\theta, \cos\theta)$ but

$$\frac{d\mathbf{t}}{ds} = (-\frac{d\theta}{ds}\sin\theta, \frac{d\theta}{ds}\cos\theta), \tag{A.1.4}$$

So $\kappa = \frac{d\theta}{ds}$ and is the rate of change the unit tangent vector makes with the axis, or equivalently, the normal vector.

In the more general case each point of $\mathcal{S}$ admits a d-dimensional tangent space spanned by vectors which we shall denote by $t^a_{\bar{\mu}}$, the $\bar{\mu}$ in this notation is just a label taking d values and *not* a contravariant index. Similarly at each point on $\mathcal{S}$ we have a $n-d$-dimensional normal space ($\mathcal{S}$ is said to have *codimension* $n-d$) which we will assume is spanned by n^a_μ, again μ is just an index labelling the vector and so takes $n-d$ values. The normal vectors are only defined on $\mathcal{S}$, it is possible to extend them off the submanifold in such a way that for all μ and ν [2]

$$n^a_\mu \nabla_a n_{\nu b} = 0, \tag{A.1.5}$$

and also so that they are orthonormal in $\mathcal{M}$, that is

$$n^a_\mu n_{\nu a} = \mu_\pm \delta_{\mu\nu}, \tag{A.1.6}$$

where,

$$\mu_\pm = \begin{cases} 1 & \text{if time-like} \\ -1 & \text{if space-like} \end{cases} \tag{A.1.7}$$

We can now define the *first fundamental form* of $\mathcal{S}$ in $\mathcal{M}$

$$h_{ab} = g_{ab} - \sum_{\mu=0}^{n-d-1} \mu_\pm n_{\mu a} n_{\mu b}, \tag{A.1.8}$$

where g_{ab} is the metric on the full space-time. This defines the projection tensor of $\mathcal{S}$. For example

$$h_{ab}h^b{}_c = h_{ac} \tag{A.1.9}$$

$$h_{ab}n^b{}_\mu = 0. \tag{A.1.10}$$

Next we can define the extrinsic curvature of $\mathcal{S}$ in $\mathcal{M}$. This object, labelled by μ, measures how the submanifold curves away from n^a_μ in $\mathcal{M}$.

$$K_{\mu ab} = h^c{}_{(a}h^d{}_{b)}\nabla_c n_{\mu d}. \tag{A.1.11}$$

Using A.1.8 we can rewrite this equation as

$$K_{\nu ab} = \left(\delta^c_a - \sum_{\mu=0}^{n-d-1} \mu_\pm n^c_\mu n_{\mu a}\right)\left(\delta^b_d - \sum_{\mu=0}^{n-d-1} \mu_\pm n^d_\mu n_{\mu b}\right)\nabla_c n_{\nu d}, \tag{A.1.12}$$

which, after using A.1.5, simplifies to

$$K_{\mu ab} = \nabla_a n_{\mu b} - \textstyle\sum_\nu \mu_\pm n_{\mu b}\left(n^d_\nu \nabla_a n_{\mu d}\right) \tag{A.1.13}$$

$$= \nabla_a n_{\mu b} - \textstyle\sum_\nu \mu_\pm \beta^\nu{}_{\mu a} n_{\nu b}, \tag{A.1.14}$$

where we have defined the *normal fundamental form* by

$$\beta^\mu{}_{\nu a} = n^d{}_\nu \nabla_a n_{\mu d}. \tag{A.1.15}$$

Note the following

- The normal forms (courtesy of the orthogonality condition) are anti-symmetric, $\beta^\mu_{\nu a} = -\beta^\nu_{\mu a}$. In particular if $n = d+1$ we get the simple relation

$$K_{ab} = \nabla_a n_b. \tag{A.1.16}$$

- If we rearrange A.1.14 to

$$\nabla_a n_{\mu b} = K_{\mu ab} + \sum_\nu \mu_\pm \beta^\nu{}_{\mu a} n_{\nu b}, \tag{A.1.17}$$

then we see that $\beta^\mu{}_{\nu a}$ can be thought of as a connection on the normal bundle.

To proceed further we define $\nabla^{(d)}_a$ to be the covariant derivative operator associated with h_{ab}. It follows that

$$\nabla^{(d)}_a T^{a_1\ldots a_k}{}_{b_1\ldots b_m} = h^{a_1}{}_{c_1}\ldots h^{d_m}{}_{b_m} h^f{}_a \nabla_f T^{c_1\ldots c_k}{}_{d_1\ldots d_m}. \tag{A.1.18}$$

This result allows us to derive equations relating the curvature of the submanifold to that of the whole space. For example

$$R^{(d)}{}_{abc}{}^d \epsilon_d = \nabla^{(d)}_a \nabla^{(d)}_b \epsilon_c - \nabla^{(d)}_b \nabla^{(d)}_a \epsilon_c, \tag{A.1.19}$$

where ϵ^a is a vector laying in $\mathcal{S}$. However using equation (A.1.18) it is straight forward to show that

$$\begin{aligned} \nabla^{(d)}_a \nabla^{(d)}_b \epsilon_c &= h^f{}_a h^d{}_b h^e{}_c \nabla_f \nabla_d \epsilon_e - h^e{}_c \left(\sum_\mu \mu_\pm K_{\mu ab} n^d \right) \nabla_d \epsilon_e + \\ & -h^d{}_b \left(\sum_\mu \mu_\pm K_{\mu ac} n^e \right) \nabla_d \epsilon_e. \end{aligned} \tag{A.1.20}$$

Now since $K_{\mu ab}$ is symmetric when we use this equation in A.1.19 the second term on the RHS must vanish, moreover the last term can be written as

$$h^d{}_b \left(\sum_\mu \mu_\pm K_{\mu ac} n^e \right) \nabla_d \epsilon_e = - \sum_\mu \mu_\pm K_{\mu ac} K_{\mu bf} \epsilon^f. \tag{A.1.21}$$

So putting it all together gives us

$$R^{(d)}{}_{abc}{}^d = h^f{}_a h^g{}_b h^k{}_c h^d{}_l R_{fgk}{}^l + \sum_\mu \mu_\pm \left(K_{\mu bc} K^d_\mu{}_a - K_{\mu ac} K^d_\mu{}_b \right). \tag{A.1.22}$$

A similar calculation also yields

$$\nabla^{(d)}_a K^a_{\mu b} - \nabla^{(d)}_b K^a_{\mu a} = R_{cd} n^d_\mu h^c{}_b. \tag{A.1.23}$$

These last two equations are known as the Gauss-Codazzi equations and are used in chapters 2 and 3.

A.2 Dirac delta function in general coordinates

Let $\mathcal{M}$ be an n-dimensional Euclidean manifold with metric

$$ds^2 = g_{ab} dx^a dx^b. \tag{A.2.24}$$

The defining property of the m-dimensional delta function, $\delta^{(m)}(\mathbf{r}-\mathbf{r}')$, is that for an arbitrary test function $f(\mathbf{r})$ we have

$$\int_{\mathbb{S}} f(\mathbf{r})\delta^{(m)}(\mathbf{r}-\mathbf{r}') = \begin{cases} f(\mathbf{r}') & \text{for } \mathbf{r}' \in \mathbb{S} \\ 0 & \text{otherwise} \end{cases} \tag{A.2.25}$$

where $\mathbb{S}$ is an open subset of $\mathcal{M}$. We can write this delta function as a product of one-dimensional functions in the following way; suppose that the set of points $\mathbf{r}' \in \mathbb{S}$ in m dimensions is multiply covered by the x_μ coordinates in (A.2.24) and that the submanifold spanned by these coordinates, denoted by $\mathcal{S}$, has the metric

$$ds^2 = h_{\mu\nu}dx^\mu dx^\nu. \tag{A.2.26}$$

Then we can write

$$\delta^{(m)}(\mathbf{r}-\mathbf{r}') = \frac{1}{\Omega}\prod_{i\neq\mu}\delta(x_i - x_i'), \tag{A.2.27}$$

where

$$\Omega = \sqrt{\hat{g}}\int_{\mathcal{S}}\sqrt{\det h_{\mu\nu}} \tag{A.2.28}$$

and $\hat{g}$ is the determinant of the metric on the space transverse to $\mathbb{S}$. Let's use this to write a two-dimensional delta function with $\mathbf{r}' = \mathbf{0}$ in plane polar coordinates as a product of one-dimensional delta functions. The metric in plane polar coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2, \tag{A.2.29}$$

where $\theta \in (0, 2\pi]$ and the origin is multiply covered by this coordinate. It now follows from (A.2.28) that

$$\Omega = 2\pi r, \tag{A.2.30}$$

and hence that

$$\delta^{(2)}(\mathbf{r}) = \frac{1}{2\pi r}\delta(r). \tag{A.2.31}$$

The generalisation from r^2 to an arbitrary function $L(r)^2$ in (A.2.29) is trivial.

Let's now see how the assumption of $\mathbb{Z}_2$ symmetry generates the one-dimensional delta function for us.

Let $g(x)$ be an even function (this is the symmetry assumption) with

$$g(0) = 0 \tag{A.2.32}$$

$$g'(0) \neq 0, \tag{A.2.33}$$

so that in a neighbourhood of the origin we can write

$$g(x) \approx |x| g'(0). \tag{A.2.34}$$

The delta function is defined by its action on test functions, consider that

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} g''(x) f(x) dx &= g'(x) f(x)|_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} g'(x) f'(x) dx \\ &\approx g'(0)(f(\epsilon) + f(-\epsilon)) - g'(0)\left(\int_0^{\epsilon} f'(x) dx - \int_{-\epsilon}^{0} f'(x) dx\right)\end{aligned}$$

and as $\lim_{\epsilon \to 0}$ the RHS becomes $2g'(0)f(0)$. On the other hand if we had integrated over a region which had not contained zero, such as $[a-\epsilon, a+\epsilon]$ then we would quickly have found that as $\epsilon \longrightarrow 0$ the integral vanishes. Motivated by this we define

$$g''(x) = 2g'(0)\delta(x). \tag{A.2.35}$$

A.3 Regularity of the Euclidean negative mode

In this section we complete the argument, following [63], that a candidate negative mode is, by dropping the $\exp(\ldots)$ factors, a physical negative mode by showing that it satisfies appropriate horizon boundary conditions.

Consider the following, general, perturbed Lorentzian metric

$$ds^2 = U(r)(1+\phi(r,z))dt^2 - V(r)^{-1}(1+\psi(r,z))dr^2 - R(r)^2(1+k(r,z))d\Omega^2 - dz^i dz_i, \tag{A.3.36}$$

where ϕ, ψ and k have z-dependence $\exp(i\mu_i z^i)$. The metric is regular at the horizon $r = r_+$ if, and only if, the perturbation is bounded as $r \longrightarrow r_+$ and

$$\phi(r_+, z) = \psi(r_+, z). \tag{A.3.37}$$

Now Wick rotate (A.3.36) (i.e. $t \longrightarrow i\tau$) and drop the $\exp(\ldots)$ factors in the ϕ, ψ and k fields. Next define a new radial coordinate near the horizon by

$$\rho = \frac{2\sqrt{r - r_+}}{\sqrt{V'(r_+)}}\left(1 + \frac{1}{2}\psi(r_+)\right), \tag{A.3.38}$$

so that the t, r part of the metric can be written as

$$d\rho^2 + \rho^2 \frac{U'(r_+)V'(r_+)}{4}(1 + \psi(r_+) - \phi(r_+))d\tau^2. \tag{A.3.39}$$

Regularity of the background solution requires $\tau \sim \tau + \beta$, the perturbation only satisfies this if

$$\phi(r_+) = \psi(r_+), \tag{A.3.40}$$

which is satisfied if equation (A.3.37) is. We conclude the correct boundary conditions are indeed satisfied.

A.4 Generality of Harmark and Obers' ansatz in the charged case

In chapter 5 we mentioned that the Harmark and Obers ansatz is consistent in the neutral case. Since the equations for the unknown functions $A(R,v)$ and $K(R,v)$ are independent of the charge, this also implies that the ansatz is consistent in the charged case. It would still be interesting, however, to ask if we can show that the *most general* solution of the equations of motion with the assumed symmetries can be written in the form (5.13,5.15) when we include charge. The following argument contains all the details for the neutral case.

We can easily show that the metric can be written in the form (5.13) by an extension of the previous argument. Starting from the 3-function conformal form (5.20), we can make the redefinitions,

$$e^B \rightarrow \bar{H}^{-\frac{d-2}{n-2}} e^B, \tag{A.4.41}$$

$$e^C \rightarrow \bar{H}^{\frac{1}{n-2}} e^B, \tag{A.4.42}$$

$$e^D \rightarrow \bar{H}^{\frac{1}{n-2}} e^D, \tag{A.4.43}$$

for any function $\bar{H}$, so that (5.20) becomes

$$ds^2 = \bar{H}^{-\frac{d-2}{n-2}} \left(e^{2B} dt^2 - \bar{H} e^{2C} \left(dr^2 + dz^2 \right) - \bar{H} e^{2D} d\Omega_{d-2}^2 \right). \tag{A.4.44}$$

If we now perform the same change of variables as was used in [85] in the neutral case,

$$R^{d-3} = R_0^{d-3} + \bar{r}^{d-3} \tag{A.4.45}$$

$$\hat{A} = f^{-1} A R_T^2 \left(\frac{\bar{r}}{R}\right)^{2(d-4)} \tag{A.4.46}$$

$$\hat{K}^{d-2} = \frac{K^{d-2}}{f} \left(\frac{\bar{r}}{R}\right)^{2(d-4)} \tag{A.4.47}$$

and then transform to the conformal form by making the further transformation,

$$\bar{r} = g(r,z), \qquad v = h(r,z) \tag{A.4.48}$$

$$\partial_r g = e^{-(d-2)k} \partial_z h, \qquad \partial_z g = e^{-(d-2)k} \partial_r h. \tag{A.4.49}$$

We can bring the metric (5.13) in the ansatz to the form (A.4.44) if

$$g^{d-3} = \frac{R_0^{d-3} e^{2B}}{1 - e^{2B}}, \tag{A.4.50}$$

and

$$e^{2a} = \frac{e^{2c}}{(\partial_r g)^2 + (\partial_z g)^2}, \qquad e^{2k} = \frac{R_T^2}{R_0^2} e^{2D} e^{\frac{2(d-5)}{(d-2)(d-3)} B} \left(1 - e^{2B}\right)^{\frac{2}{d-3}}. \tag{A.4.51}$$

The system of equations in (A.4.49) imply an integrability condition which together with (A.4.51) imply that

$$(\partial_r^2 + \partial_z^2) B + (\partial_r B)^2 + (\partial_z B)^2 + (d-2)(\partial_r B \partial_r D + \partial_z B \partial_z D) = 0, \tag{A.4.52}$$

the same integrability condition we had in the neutral case.

If we assume that the arbitrary function $\bar{H}$ introduced in the redefinitions (A.4.41) is identified with the dilaton as in (5.15), i.e., $e^{a\phi} = \bar{H}^2$, we can show that this integrability condition is again implied by the equations of motion. The most general form for $F_{\mu\nu}$ consistent with the assumed symmetries has only F_{tz} and F_{tr} non-zero; hence we can write

$$F_{\lambda t} F^{\lambda}_{\ t} = \frac{1}{2} g_{tt} F^2, \tag{A.4.53}$$

and so eliminating the dilation field using (5.40) from the t,t component of the graviton equation (5.39) gives us

$$R_{tt} = g_{tt} \frac{n-3}{2(n-2)} \nabla^2 (\ln \bar{H}). \tag{A.4.54}$$

This equation reduces to exactly (A.4.52) for the metric given in equation (A.4.44) and therefore we conclude that this integrability condition is implied by the equations of motion.

However, this is not yet enough to show that the general solution takes the form (5.13,5.15): we have not yet shown that $\bar{H} = H(R)$, and we have no coordinate freedom left to redefine it. The problem can be simply stated in coordinate-independent terms: in the charged case, there are two a priori independent scalar quantities, namely the norm of the time-like Killing vector ∂_t and the dilaton. The ansatz (5.13,5.15) assumes a specific functional form for both of these. While we can choose coordinates so that one of them takes the specified form, it will not be possible to do this for both of them in general, without using some additional information.

Thus, while it seems quite natural to us to assume that the ansatz (5.13,5.15) describes the most general solution of the equations of motion in the charged case as well, we cannot show this by some analogue of the arguments in [84, 85]. Rather, verifying our belief would require explicitly solving the equations of motion. We reiterate that this question of generality is irrelevant to the argument in the body of the argument, which required only the observation that uncharged solutions of the form (5.13,5.15) lift to charged solutions.

Appendix B

Perturbation details

B.1 Perturbation equations

The Einstein-Gauss-Bonnet equations are:

$$G_{ab} + H_{ab} = T_{ab}, \tag{B.1.1}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \tag{B.1.2}$$

whereas the contribution from the Gauss-Bonnet term reads,

$$H_{ab} = \alpha\Big[-\frac{1}{2}g_{ab}(R^2 - 4R^{cd}R_{cd} + R^{cdef}R_{cdef}) \tag{B.1.3}$$

$$+2RR_{ab} - 4R_{ac}R_b{}^c - 4R^c{}_{adb}R_c{}^d + 2R_{acde}R_b{}^{cde}\Big]. \tag{B.1.4}$$

In fact we will write the equations in the following way

$$R_{ab} - \alpha\Big[\frac{1}{2-n}g_{ab}(R^2 - 4R^{cd}R_{cd} + R^{cdef}R_{cdef}) + 2RR_{ab} - 4R_{bc}R_a{}^c + \\ - 4R^d{}_{acb}R_d{}^c + 2R_{acde}R_b{}^{cde}\Big] = T_{ab} + \frac{1}{2-n}g_{ab}T, \tag{B.1.5}$$

and consider a metric perturbation of the form (from now on lower case Latin letters will run over all dimensions)

$$g_{ab} \rightarrow g_{ab} + h_{ab}, \tag{B.1.6}$$

together with an additional energy-momentum perturbation whose components are $\overline{\delta T}_{ab}$.

The linearised Riemann tensor is calculated to be,

$$2\delta R^{a}{}_{bcd} = g^{ap}\left(-\nabla_c\nabla_p h_{bd} - \nabla_d\nabla_b h_{cp} + \nabla_c\nabla_b h_{dp} + \nabla_d\nabla_p h_{bc} - R^{l}{}_{bcd}h_{lp} - R^{l}{}_{pcd}h_{bl}\right) \tag{B.1.7}$$

from which we obtain the Lichnerowicz operator,

$$2\delta R_{ab} = -\nabla^2 h_{ab} - 2R_{cadb}h^{cd} + 2R_{c(a}h_{b)}^{\ c} + \frac{1}{2}\nabla_b(2\nabla_c h^c_a - \nabla_a h) + \frac{1}{2}\nabla_a(2\nabla_c h^c_b - \nabla_b h), \tag{B.1.8}$$

also,

$$\delta R = g^{ab}\delta R_{ab} - h^{ab}R_{ab}. \tag{B.1.9}$$

Let's perturb the energy-momentum part of equation B.1.1,

$$\delta T_{ab} = \left(\delta\hat{T}_{ab} - \frac{1}{2}\left(h^r_r + h^\theta_\theta\right)\hat{T}_{ab}\right)\delta^2(r), \tag{B.1.10}$$

$$\delta T = -h^{ab}T_{ab} + g^{ab}\delta T_{ab}. \tag{B.1.11}$$

So if we have $\hat{T}_{\mu\nu} = \Delta g_{\mu\nu}$ where Δ is the angular defect of the background space-time (which is the situation we are interested in) then we find that,

$$\delta\left(T_{\mu\nu} + \frac{1}{2-n}g_{\mu\nu}T\right) = \overline{\delta T}_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right). \tag{B.1.12}$$

Similarly we also have,

$$\delta\left(T_{rr} + \frac{1}{2-n}g_{rr}T\right) = g_{rr}\frac{\Delta}{2}\left(h^\theta_\theta - h^r_r\right)\delta^2(r) + \overline{\delta T}_{rr} + $$
$$+ \frac{1}{2-n}g_{rr}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right), \tag{B.1.13}$$

$$\delta\left(T_{\theta\theta} + \frac{1}{2-n}g_{\theta\theta}T\right) = g_{\theta\theta}\frac{\Delta}{2}\left(h^r_r - h^\theta_\theta\right)\delta^2(r) + \overline{\delta T_{\theta\theta}} + $$
$$+ \frac{1}{2-n}g_{\theta\theta}\left(\overline{\delta T} + g^{rr}\overline{\delta T}_{rr} + g^{\theta\theta}\overline{\delta T}_{\theta\theta}\right), \tag{B.1.14}$$

$$\delta\left(T_{r\theta} + \frac{1}{2-n}g_{r\theta}T\right) = \overline{\delta T}_{r\theta} - \Delta h_{r\theta}\delta^2(r), \tag{B.1.15}$$

and the structure of the other cross terms is the same as the one above. To calculate the equations for the LHS of B.1.1 we need the perturbations of the separate parts of the Gauss-Bonnet term. These can be written

$$\begin{aligned}
\delta\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right) &= 2R\delta R - 8(\delta R_{ab})R^{ab} + 8h^{ac}R_{ab}R^b_c + \\
&+ 2(\delta R^p{}_{abc})R_p{}^{abc} + \\
&+ R^a{}_{pst}\left(R^p{}_{ab}{}^t h^{bs} + R^p{}_a{}^s{}_c h^{ct}\right), \qquad \text{(B.1.16)}
\end{aligned}$$

$$\begin{aligned}
\delta\left(R^k{}_{mpn}R^p_k\right) &= \delta\left(R^k{}_{mpn}\right)R^p_k + R^k{}_m{}^s{}_n\delta R_{ks} - R^k{}_{mpn}R_{ks}h^{ps}, &\text{(B.1.17)}\\
\delta\left(R_{mp}R^p_n\right) &= (\delta R_{mp})R^p_n + (\delta R_{nq}R^q_m - h^{pq}R_{mp}R_{nq}), &\text{(B.1.18)}\\
\delta\left(R_{mqsp}R_n{}^{qsp}\right) &= \delta(R^q{}_{msp})R^{sp}{}_{qn} + \delta(R^s{}_{rqn})R^q{}_{ms}{}^r - R^q{}_{msp}R^s{}_{rqn}h^{rp}, &\text{(B.1.19)}
\end{aligned}$$

Of course they can be be put together to form the full equations, however having this written out explicitly in this generality isn't useful in any way.

Now let's suppose we have a background metric of the form

$$ds^2 = dt^2 - dx^2 - \ldots - dr^2 - L(r)^2 d\theta^2, \qquad \text{(B.1.20)}$$

which is the example we consider in this thesis.

The non-zero components of the Riemann tensor are

$$\begin{aligned}
R^r{}_{\theta r\theta} &= -LL'', &\text{(B.1.21)}\\
R^\theta{}_{r\theta r} &= -\frac{L''}{L}, &\text{(B.1.22)}
\end{aligned}$$

and the non-zero Ricci tensor components

$$\begin{aligned}
R^r{}_r &= \frac{L''}{L}, &\text{(B.1.23)}\\
R^\theta{}_\theta &= \frac{L''}{L}. &\text{(B.1.24)}
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right) =& \frac{L''}{L}\left(4\delta R - 8g^{rr}\delta R_{rr} - 8g^{\theta\theta}\delta R_{\theta\theta}\right) + &\text{(B.1.25)}\\
&- 4\left(\frac{L''}{L}\right)^2\left(h^r_r + h^\theta_\theta\right) + &\text{(B.1.26)}\\
&+ 4(\delta R^r{}_{\theta r\theta})R_r{}^{\theta r\theta} + 4(\delta R^\theta{}_{r\theta r})R_\theta{}^{r\theta r}, &\text{(B.1.27)}
\end{aligned}$$

and for the other $\mu\nu$ components we find

$$\begin{aligned}
\delta\left(RR_{\mu\nu}\right) &= 2\frac{L''}{L}\delta R_{\mu\nu} && \text{(B.1.28)}\\
\delta\left(R_{\mu a}R_{\nu}{}^{a}\right) &= 0 && \text{(B.1.29)}\\
\delta\left(R^{a}{}_{\mu b\nu}R_{a}{}^{b}\right) &= \frac{L''}{L}\left(\delta R^{r}{}_{\mu r\nu}+\delta R^{\theta}{}_{\mu\theta\nu}\right) && \text{(B.1.30)}\\
\delta\left(R_{\mu abc}R_{\nu}{}^{abc}\right) &= 0. && \text{(B.1.31)}
\end{aligned}$$

After the first part of a lengthy calculation we can then write the $\mu\nu$ components of the perturbation equations as

$$\begin{aligned}
&\delta R_{\mu\nu}+ && \text{(B.1.32)}\\
&+\alpha\frac{L''}{L}\Big(-\frac{1}{4}g_{\mu\nu}(4g^{\lambda\rho}\delta R_{\lambda\rho}+2(\bar\nabla^2 h^r_r+\bar\nabla^2 h^\theta_\theta)-2\big(\partial_r^2 h^r_r-\frac{1}{L^2}\partial_\theta^2 h^r_r+\\
&+3\frac{L'}{L}\partial_r h^r_r-3\frac{L'}{L}\partial_r h^\theta_\theta-\partial_r^2 h^\theta_\theta+\frac{1}{L^2}\partial_\theta^2 h^\theta_\theta+\frac{4}{L^2}\partial_r\partial_\theta h_{r\theta}+2\frac{L''}{L}\boxed{(h^r_r-h^\theta_\theta)}\big)\Big)+\\
&+4\delta R_{\mu\nu}+\\
&+2\big(\partial_r\partial_\mu h_{\nu r}+\partial_\nu\partial_r h_{\mu r}-\partial_r^2 h_{\mu\nu}-\partial_\mu\partial_\nu h_{rr}+\frac{1}{L^2}\partial_\theta\partial_\mu h_{\nu\theta}+\frac{L'}{L}\partial_\mu h_{\nu r}+\\
&+\frac{1}{L^2}\partial_\nu\partial_\theta h_{\mu\theta}+\frac{L'}{L}\partial_\nu h_{\mu r}-\frac{1}{L^2}\partial_\theta^2 h_{\mu\nu}-\frac{L'}{L}\partial_r h_{\mu\nu}-\frac{1}{L^2}\partial_\mu\partial_\nu h_{\theta\theta})\big)+\\
&+\alpha\frac{L''}{L}\Big(-\frac{1}{4}g_{\mu\nu}(4\partial_r\partial_a h^a_r+4\frac{L''}{L}\boxed{(h^r_r-h^\theta_\theta)}-4\left(\frac{L'}{L}\right)^2(h^r_r-h^\theta_\theta)+ && \text{(B.1.33)}\\
&+4\frac{L'}{L}\partial_r\left(h^r_r-h^\theta_\theta\right)-2\partial_r^2 h+\\
&+\frac{4}{L^2}\partial_\theta\left(\partial_a h^a_\theta+LL'h^\theta_r\right)+4\frac{L'}{L}\left(\partial_a h^a_r+\frac{L'}{L}(h^r_r-h^\theta_\theta)\right))-\frac{2}{L^2}\partial_\theta^2 h-2\frac{L'}{L}\partial_r h)\Big).
\end{aligned}$$

We have split the equation into two parts, the second part begins at B.1.33. If we calculate the equations with a harmonic gauge choice then all the terms in the second part are zero (with this choice also understood in $\delta R_{\mu\nu}$), this leaves troublesome terms with $(L''/L)^2$ factors which without the harmonic gauge choice cancel (such terms have been boxed for clarity). We therefore chose to work with a Gaussian normal gauge in which

$$h_{r\mu}=0,\quad h_{r\theta}=0 \text{ and } h_{rr}=0. \qquad \text{(B.1.34)}$$

With this the full equation simplifies somewhat to

$$\begin{aligned}
\delta R_{\mu\nu}-\alpha\frac{L''}{L}\left(g_{\mu\nu}(\bar\nabla^2 h^4-\partial_\mu\partial^\lambda h^\mu_\lambda)-2\bar\nabla^2 h_{\mu\nu}-2\partial_\mu\partial_\nu h^4+\right.&\\
\left.+2(\partial_\mu\partial_\lambda h^\lambda_\nu+\partial_\nu\partial_\lambda h^\lambda_\mu)\right)=\overline{\delta T_{\mu\nu}}-\frac{1}{4}g_{\mu\nu}\overline{\delta T}, & \qquad \text{(B.1.35)}
\end{aligned}$$

where

$$\delta R_{\mu\nu} = -\frac{1}{2}\bar{\nabla}^2 h_{\mu\nu} + \frac{1}{2}\partial_r^2 h_{\mu\nu} + \frac{1}{2L^2}\partial_\theta^2 h_{\mu\nu} + \frac{L'}{2L}\partial_r h_{\mu\nu} + \\ +\frac{1}{2}\left(\partial_\mu \partial_a h^a_\nu + \partial_\nu \partial_a h^a_\mu\right) - \frac{1}{2}\partial_\mu\partial_\nu h. \tag{B.1.36}$$

The rr and $\theta\theta$ perturbation equations can be written respectively as

$$\delta R_{rr} + \frac{1}{4}\alpha\delta\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right) = \delta\Delta - \frac{\Delta}{2}\left(h^\theta_\theta - h^r_r\right)\delta^2(r), \tag{B.1.37}$$

$$\delta R_{\theta\theta} - \frac{1}{4}g_{\theta\theta}\alpha\delta\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right) = g_{\theta\theta}\left(\delta\Delta - \frac{\Delta}{2}\left(h^\theta_\theta - h^r_r\right)\delta^2(r)\right), \tag{B.1.38}$$

where

$$\delta R_{rr} = -\frac{1}{2}\bar{\nabla}^2 h_{rr} + \frac{1}{2L^2}\partial_\theta^2 h_{rr} - \frac{L'}{2L}\partial_r h_{rr} - \frac{1}{L^2}\partial_r\partial_\theta h_{r\theta} - \frac{L'}{L}\partial_r h^\theta_\theta + \partial_r\partial_\mu h^\mu_r - \frac{1}{2}\partial_r^2 h \tag{B.1.39}$$

and

$$\delta R_{\theta\theta} = -\frac{1}{2}\bar{\nabla}^2 h_{\theta\theta} + \frac{1}{2}\partial_r^2 h_{\theta\theta} - \frac{3}{2}\frac{L'}{L}\partial_r h_{\theta\theta} + \frac{1}{2L^2}\partial_\theta^2 h_{\theta\theta} + 2\frac{L'}{L}\partial_\theta h_{r\theta} + (L')^2 h_{rr} + \\ + LL'' h^r_r + \left(\frac{L'}{L}\right)^2 h_{\theta\theta} + \partial_\theta\left(\partial_a h^a_\theta + LL' h^\theta_r\right) + \frac{L'}{L}\left(\partial_a h^a_r + \frac{L'}{L}\left(h^r_r - h^\theta_\theta\right)\right) + \\ - \frac{1}{2}\partial_\theta^2 h - \frac{1}{2}\frac{L'}{L}\partial_r h. \tag{B.1.40}$$

The important point for the analysis in chapter 4 is that $\delta R_{\theta\theta}$ contains an L'' term which is not part of the α corrections (even if we are careful to consider such terms hidden in $\partial_r^2 h_{\theta\theta}$). The equations in the Gaussian normal gauge obtained from these are written out in the main text.

All the other components of the equations have no α contributions, in the Gaussian normal system they are also written out explicitly in chapter 4.